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E. B. Nasatyr and B. Steer

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The Narasimhan–Seshadri theorem for parabolic bundles: an orbifold approach

By E. B. Nasatyr[1]† and B. Steer[2]

[1] *Peterhouse, Cambridge CB2 1RD, UK*

[2] *Hertford College, Oxford OX1 3BW, UK*

Given a stable parabolic bundle over a Riemann surface, we study the problem of finding a compatible Yang–Mills connexion. When the parabolic weights are rational there is an equivalent problem on an orbifold bundle. When the weights are irrational our method is to choose a sequence of approximating rational weights, obtain a corresponding sequence of Yang–Mills connexions on the resulting orbifold bundles and obtain the solution as the limit of this sequence: we need to consider mildly singular connexions which locally about a marked point take the form $d - \lambda \mathrm{i} d\theta + a$. Here λ is a constant diagonal matrix whose entries depend on the weights and their rational approximations, $\theta = \arg(z)$ for z a local uniformizing (orbifold) coordinate centred on the marked point and a is an L^2_1 connexion matrix. In this context we find all the necessary gauge-theoretic tools to prove the theorem, including a version of Uhlenbeck's weak compactness theorem, provided $|\lambda|$ is sufficiently small. (One of the advantages of this approach is that we do analysis on a *compact* orbifold rather than on the punctured surface.) Our methods also allow us to consider the analogous problem for stable parabolic Higgs bundles.

1. Introduction

In this paper we prove a version of the Narasimhan–Seshadri theorem for stable parabolic bundles. The correspondence between stable parabolic bundles and representations of the fundamental group of the punctured surface (with prescribed holonomy around the punctures) is due to Mehta & Seshadri (1980) (though only in the case $g \geqslant 2$). Analytical proofs in terms of Yang–Mills connexions have been obtained recently, first by Biquard (1991, théorème 2.5) and subsequently by Poritz (1993); however, the theorem admits conjectural generalizations, e.g. to higher dimensions (Kronheimer & Mrowka 1993), with important potential applications and we hope that our proof will offer new insights into these problems. We also follow Hitchin (1987) and Simpson (1988, 1990) (as well as the preprint of Konno (1992), of which we have only recently become aware) in considering the extension of the problem to stable parabolic Higgs bundles.

Let M be a Riemann surface and let $F \to M$ be a smooth complex vector bundle. We say that F is a *weighted bundle* if 'weighted flags' are given in the fibres F_{p_i} for a finite set of 'marked points' $p_1, \ldots, p_d \in M$. The addition of a holomorphic structure

† Present address: Department of Mathematical Sciences, University of Aberdeen, Edward Wright Building, Dunbar Street, Old Aberdeen AB9 2TY, UK.

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on F makes it a *parabolic bundle*. There are notions of weighted or parabolic degree and of stability for parabolic bundles, involving the weighted flags. These ideas are due to Mehta & Seshadri (1980); see § 2 for further details.

Now suppose that F is given a Hermitian metric and M a Riemannian metric. Then the Narasimhan–Seshadri theorem (in a version due to Donaldson (1983)) gives, on F, a correspondence between stable holomorphic structures (modulo automorphisms) and irreducible Yang–Mills connexions (modulo gauge transformations). In the context of weighted bundles and parabolic structures it is not immediately clear what notions of 'weighted Hermitian metric' and 'weighted unitary connexion' are necessary to generalize the theorem. For instance, since the parabolic degree is not, in general, an integer weighted unitary connexions must certainly be singular at the marked points for a Chern–Weil formula to hold. Moreover, according to Mehta & Seshadri, stable parabolic structures correspond to unitary representations of $\pi_1(M \setminus \{p_1, \ldots, p_d\})$ with the value on small loops around the punctures determined by the weights so that our weighted unitary connexions should have holonomy around the marked points and our solutions be projectively flat. Appropriate definitions were given in Biquard (1991) and are given in § 2: a weighted Hermitian metric is degenerate at a marked point; a weighted unitary connexion has smooth $(0,1)$-part but singular $(1,0)$-part there.

We can now state our main theorem as follows.

Theorem 2.1. *Suppose that M is a Riemann surface with a Riemannian metric and that $F \to M$ is a Hermitian weighted bundle with a given stable parabolic structure. Then there exists a weighted automorphism of F, unique modulo weighted gauge transformations, taking the weighted connexion of the parabolic structure to a Yang–Mills weighted connexion.*

We are also able to show that near a marked point a Yang–Mills weighted connexion takes a particularly simple form in a radial gauge (proposition 2.2).

Now, when the weights of F are rational there is an orbifold bundle $\tilde{F}$ (see § 3 *a* for a brief discussion of orbifold bundles or 'V-bundles') with orbifold structure at the marked points such that stable parabolic structures on F correspond to stable holomorphic structures on $\tilde{F}$. (The construction of $\tilde{F}$ from F proceeds via a holomorphic clutching around the marked points.) Moreover, the Narasimhan–Seshadri theorem for V-bundles is comparatively straightforward. These results are originally due to Furuta & Steer (1992) (independently to Boden (1991)) and are discussed in § 3 *b*. Thus, in the case when the weights are rational, stable parabolic structures on F correspond to Yang–Mills connexions on the associated V-bundle $\tilde{F}$. However, this doesn't solve the problem of producing a Yang–Mills weighted connexion on F (the passage from F to $\tilde{F}$ is singular at the marked points) nor of dealing with the case of irrational weights. (In the case when the weights are irrational, Mehta & Seshadri showed that for many purposes it suffices to consider a set of sufficiently close rational weights to study the moduli of stable parabolic structures on F. Nevertheless, the rational case is certainly *not* sufficient, e.g. to discuss the total moduli space when the weights are allowed to vary.)

The main idea of our proof is that when the weights of F are irrational we can still consider a V-bundle $\tilde{F}$ (constructed by a weighted unitary clutching construction) associated to a set of sufficiently close rational weights. Now we will have to consider connexions on $\tilde{F}$ with mild singularities at the orbifold points: we define a *connexion* *with model singularity* on $\tilde{F}$ to be one which is locally like

$$d - \begin{pmatrix} \epsilon_1 & & 0 \\ & \ddots & \\ 0 & & \epsilon_l \end{pmatrix} \mathrm{i}d\theta + a, \tag{1}$$

where the ϵ_is are determined by the differences between the given irrational weights and their rational approximations and a is a smooth (or, more generally, L_1^2) endomorphism-valued 1-form. (Here $\theta = \arg(z)$, where z is a local uniformizing, or orbifold, coordinate centred on the marked point.) Eventually we will also allow the ϵ_is to vary. The details of this are in § 4 *a*.

There is a simple idea which allows us to do analysis despite these model singularities: although $\mathrm{d}\theta$ is singular at the marked point, if the off-diagonal terms of the connexion matrix a have sufficient vanishing there then the operator $a \mapsto [a, \mathrm{d}\theta]$ is bounded from L_1^2 to L^2 and we can estimate its norm. Our connexions will satisfy these vanishing conditions because of the orbifold structure. Thus, for instance, we can show that the curvature of (1) is L^2. Analogous arguments show that the usual gauge-theoretic machinery applies to these connexions with model singularity: the relevant results, including versions of Uhlenbeck's weak compactness theorem, are proved in § 4 *b* and § 4 *c*. (In Biquard (1991) the sophisticated machinery of Lockhart & McOwen (1985) is invoked to deal with the analytical problems posed by the singularities at the marked points but the use of V-bundles allows us to use the more elementary arguments just outlined—in particular we effectively work on compact surfaces and use ordinary rather than weighted Sobolev spaces.)

Algebraic-geometric information coming from the stability of the parabolic structure is used to prove that the solution lies in the orbit we started with. In § 5 *a* we show that, although the $\overline{\partial}$-operators of connexions like (1) on $\tilde{F}$ are singular, they correspond to holomorphic structures on the weighted bundle F. A comparison of orbifold and smooth Riemannian metrics and a discussion of local forms of Yang–Mills connexions around marked points is contained in § 5 *b*: the main result is that in a radial gauge, the Yang–Mills equation together with equivariance conditions determine the connexion completely; proposition 5.6. This is unsurprising as in two real dimensions Yang–Mills connexions are projectively flat and hence, up to twisting by orbifold line-bundles, locally trivial. This radial gauge-fixing helps us to simplify the proof of the main theorem but is not essential; for Higgs bundles one gets slightly weaker results and we expect that something similar is true in higher dimensions.

At this point all the preliminaries are in place and the theorem is proved in § 5 *c*. For the proof we take a sequence of rational weights approximating the given irrational weights. Each set of rational weights gives us a solution on a certain orbifold bundle, by the orbifold Narasimhan–Seshadri theorem. Transferring each of these connexions to $\tilde{F}$ we get a sequence of Yang–Mills connexions with varying model singularities and by an appropriate version of Uhlenbeck's weak compactness theorem (proposition 4.10) we get weak convergence of a subsequence to a connexion of the required form. The explicit description of the connexion in a radial gauge tells us that this connexion transfers smoothly back to F and the resulting connexion is in the original orbit by stability.

Finally, § 6 deals with the analogous problem for parabolic Higgs bundles. The final result is theorem 2.3: in contrast to the case when there is no parabolic Higgs field we cannot use radial gauge-fixing and the solution weighted automorphism is only continuously differentiable, not smooth, at the marked points.

Let us briefly recap the contents of the sections. The basic definitions—of parabolic and weighted bundles, weighted Hermitian metrics, weighted unitary connexions and parabolic Higgs bundles—are given in § 2 together with the statement of the main results (theorem 2.1, proposition 2.2 and theorem 2.3). The subject of § 3 is the correspondence between parabolic bundles with rational weights and V-bundles: V-bundles are introduced in § 3 *a* and the correspondence itself discussed in § 3 *b*.

The core of this paper is § 4 where the problem is transferred to the V-bundle $\tilde{F}$. The construction of $\tilde{F}$ and the definition of a connexion with model singularity are given in § 4 *a*. In § 4 *b* the discussion is widened to allow Sobolev spaces of connexions with model singularity to be considered: the main estimates which show how the model singularity is 'controlled' by the vanishing conditions at the orbifold points and most of the usual gauge-theoretic machinery follow, with the weak compactness results in § 4 *c*.

The penultimate section, § 5, contains the proof of the main theorem: § 5 *a* gives preliminary results on the $\bar{\partial}$-operator associated to a connexion with model singularity, § 5 *b* is concerned with a comparison of orbifold and smooth Riemannian metrics and local forms of Yang–Mills connexions around marked points and the proof of theorem 2.1 is given in § 5 *c*. Finally, § 6 deals with parabolic Higgs bundles.

2. Parabolic bundles: definitions and statement of results

In this section we briefly review our basic definitions and state our main theorems. Throughout this paper M denotes a closed, connected Riemann surface equipped with a (finite, non-zero) number of distinguished (or 'marked') points. For convenience we shall often assume that there is a single marked point $p \in M$ (our methods generalize immediately to more than one marked point but the notation is simplified by assuming only one).

First we recall the definition of a 'parabolic bundle' (Mehta & Seshadri 1980) and introduce the weaker notion of a 'weighted bundle.' We give an appropriate definition of a Hermitian metric on a weighted bundle (cf. Biquard 1991) and of a 'weighted unitary connexion'.

Let $\mathcal{F}$ be a holomorphic structure on an underlying smooth bundle $F \to M$. A *quasi-parabolic structure* on $\mathcal{F}$ is a flag of length $m \geqslant 1$ in the fibre F_p;

$$F_p = F_1 \underset{\neq}{\supset} F_2 \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} F_m \underset{\neq}{\supset} F_{m+1} = 0. \tag{2}$$

A quasi-parabolic bundle $\mathcal{F}$ is a *parabolic bundle* if there are 'weights' attached to the quasi-parabolic structure; that is, attached to the flag there is a sequence of weights, $0 \leqslant \lambda'_1 < \lambda'_2 < \cdots < \lambda'_m < 1$, where m is the length of the flag. For $j = 1, \ldots, m$, let $\mu_j = \dim(F_j/F_{j+1})$, the 'multiplicity' of the weight λ'_j. It is convenient to write $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_l < 1$ for the weights repeated according to their multiplicities.

The *parabolic degree* of a parabolic bundle $\mathcal{F}$ is defined by

$$\mathrm{ParDeg}\,(F) = c_1(F) + \sum_{j=1}^{m} \mu_j \lambda'_j = c_1(F) + \sum_{i=1}^{l} \lambda_i$$

(where $c_1(F)$ is the first Chern class, interpreted as an integer degree).

A basis $\{e_1, \ldots, e_l\}$ for the fibre at the marked point is said to *respect the quasi-parabolic structure* if $\{e_{\mu_1+\cdots+\mu_{m-1}+1}, \ldots, e_l\}$ span F_m, $\{e_{\mu_1+\cdots+\mu_{m-2}+1}, \ldots, e_l\}$ span F_{m-1}, and so on. An endomorphism ψ of a parabolic bundle is a *parabolic endomorphism* if it respects the quasi-parabolic structure at the marked point. The condition that ψ respects the quasi-parabolic structure at p amounts to saying that ψ_p is given by a block lower-triangular matrix with respect to any basis which respects the quasi-parabolic structure; i.e. if $\lambda_i < \lambda_j$ then $(\psi_p)_{ij} = 0$. (Note that this depends only on the quasi-parabolic structure.)

A holomorphic subbundle $\mathcal{F}'$ of a parabolic bundle $\mathcal{F}$ is naturally a *parabolic subbundle*; i.e. it naturally inherits a parabolic structure by intersecting the flag (2) with F'_p and taking the corresponding weights. Of course the resulting flag may not be strictly descending; in this case we define the flag using only those $F'_p \cap F_j$ $(j = 1, \ldots, m)$ for which $F'_p \cap F_j \supset_{\neq} F'_p \cap F_{j+1}$.

Having defined parabolic subbundles and degrees we have the usual definitions of stable and semi-stable parabolic bundles.

Parabolic bundles are the basic objects which we want to study but we want to consider many holomorphic structures on the underlying 'bundle with weighted flag' simultaneously; therefore we introduce the notion of a 'weighted bundle' as follows. Define a *weighted flag structure* on $F \to M$ to be exactly the data at the marked point required for a parabolic bundle (i.e. a flag (2) with associated weights is required but *not* a holomorphic structure). Then F equipped with a weighted-flag-structure is a *weighted bundle*. The *weighted degree* $\mathrm{WeiDeg}\,(F)$ is defined in exactly the same way as the parabolic degree. A *weighted endomorphism* of a weighted bundle is just a smooth endomorphism of the underlying bundle which respects the flag (i.e. a weighted endomorphism has the same property—independent of the weights—at the marked point as a parabolic endomorphism but is not necessarily holomorphic).

Let F be a weighted bundle of rank l with weights $0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_l < 1$. We define a *weighted Hermitian metric* to be a Hermitian metric over the complement of the marked point which, in a neighbourhood of the marked point, has the form

$$g^* \begin{pmatrix} |w|^{2\lambda_1} & & 0 \\ & \ddots & \\ 0 & & |w|^{2\lambda_l} \end{pmatrix} g, \tag{3}$$

for some weighted automorphism g and in coordinates which respect the flag structure. (Here w is the local holomorphic coordinate and g^* is the conjugate-transpose of g.) We stress that the quadratic form (3) extends smoothly over the marked point; it is merely degenerate there. We call a weighted bundle (respectively, a parabolic bundle) with a weighted Hermitian metric a *Hermitian weighted bundle* (respectively, a *Hermitian parabolic bundle*). (Notice that this concept does not depend on the particular choice of local holomorphic coordinate, w.) Define a *weighted gauge transformation* to be a weighted automorphism which preserves the weighted Hermitian metric. Denote the weighted gauge transformations by $\mathcal{G}_{\mathrm{wei}}(F)$ and the weighted automorphisms by $\mathcal{G}^{\mathrm{c}}_{\mathrm{wei}}(F)$.

We can always find a *weighted unitary frame* (with respect to w), i.e. a frame over a neighbourhood of the marked point where the weighted Hermitian metric has the 'standard form',

$$h_0(w) = \begin{pmatrix} |w|^{2\lambda_1} & & 0 \\ & \ddots & \\ 0 & & |w|^{2\lambda_l} \end{pmatrix}.$$

This notion clearly depends on the particular choice of w. (Genuine unitary frames exist over the complement of the marked point but we shan't use them.)

We define a *weighted unitary connexion* to be a connexion which is smooth and unitary over the complement of the marked point and such that the associated $\bar{\partial}$-operator extends smoothly over the marked point. Of course, this definition automatically gives us the usual correspondence (the 'Chern correspondence') between connexions and holomorphic structures in the presence of a Hermitian metric. Notice that we do not attempt to describe the singularities in the $(1,0)$-part of the connexion (since we are not going to be doing analysis on parabolic bundles). It might be thought that the Gram–Schmidt process could be used to construct 'nice' frames in which these singularities take 'model' forms but the Gram–Schmidt process fails for weighted Hermitian metrics.

We also note that weighted unitary connexions have the two required properties mentioned in the introduction: there is a Chern–Weil formula (Biquard 1991, proposition 2.9) and the holonomy around the marked point is determined by the weights.

We say that a weighted unitary connexion A is *Yang–Mills* if its curvature satisfies $F_A = -2\pi \mathrm{i} \operatorname{ParDeg}(F)(*1)I$, where '$*$' denotes the Hodge star of a given Riemannian metric on M. As in the standard case we look for a Yang–Mills weighted unitary connexion in the orbit of a stable parabolic structure. Now we state the main theorem.

Theorem 2.1. *Suppose that M is a Riemann surface with a Riemannian metric and that $F \to M$ is a Hermitian weighted bundle with a given stable parabolic structure. Then there exists a weighted automorphism of F, unique modulo weighted gauge transformations, taking the weighted connexion of the parabolic structure to a Yang–Mills weighted connexion.*

This is essentially Biquard's theorem (Biquard 1991, théorème 2.5). However, in the course of the proof we also show that the Yang–Mills weighted connexion can also be supposed to take a particularly simple form about the marked point. This theorem, and the proposition below, are proved in § 5.

Proposition 2.2. *Let $F \to M$ be a Hermitian weighted bundle and let A be a weighted unitary connexion on F which is Yang–Mills with respect to a Riemannian metric on M, locally like $\mathrm{d}w \otimes \mathrm{d}\bar{w}$ near the marked point. Then there is a weighted gauge transformation, g, such that $g(A)$ has the form*

$$g(A) = d + \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_l \end{pmatrix} \frac{\mathrm{d}w}{w} - \tfrac{1}{4}\pi\mu(F)I(\bar{w}\mathrm{d}w - w\mathrm{d}\bar{w})$$

locally about the marked point, in a weighted unitary frame.

A *parabolic Higgs field* on a parabolic bundle $\mathcal{F}$ is a $(1,0)$-form valued endomorphism of $\mathcal{F}$, holomorphic away from the marked point and locally of the form

$$\phi_{ij}(w)\mathrm{d}w/w \quad \text{if } \lambda_i > \lambda_j \quad \text{and} \quad \phi_{ij}(w)\mathrm{d}w \quad \text{if } \lambda_i \leqslant \lambda_j,$$

where the ϕ_{ij}s are holomorphic in a holomorphic frame which respects the quasi-parabolic structure at the marked point. Notice that a parabolic Higgs field is then a holomorphic parabolic endomorphism with values in $\Omega^{1,0}(\log r)$, the sheaf of $(1,0)$-forms generated by $\mathrm{d}w/w$: in the language of Simpson (1990) this is a filtered regular Higgs field (though of a very particular type).

A parabolic bundle together with a parabolic Higgs field is called a *parabolic Higgs bundle*. Stability of parabolic Higgs bundles is defined with reference to the ϕ-invariant subbundles only. (See Hitchin (1987) for the original work on Higgs bundles over Riemann surfaces and Nasatyr & Steer (1995) for the extension to orbifold Riemann surfaces.)

If F is a Hermitian weighted bundle, A a weighted unitary connexion on F and ϕ a parabolic Higgs field, holomorphic in the sense that $\overline{\partial}_A\phi = 0$, then (A, ϕ) is called a *weighted Higgs pair*. A weighted Higgs pair is said to be *Yang–Mills-Higgs* if $F_A + [\phi, \phi^*]$ is equal to the Yang–Mills curvature. (Here ϕ^* is defined as the combination of taking the adjoint of an endomorphism with respect to the weighted Hermitian metric and taking the complex conjugate of a $(1,0)$-form.)

Our main theorem on the existence of Yang–Mills–Higgs pairs is as follows. Notice that for this theorem we need to consider weighted unitary connexions and parabolic Higgs fields which are obtained by the action of a weighted automorphism which is only C^1 at the marked point.

Theorem 2.3. *Suppose that M is a Riemann surface with a Riemannian metric and that $F \to M$ is a Hermitian weighted bundle with a given stable parabolic Higgs structure. Then there exists a weighted automorphism of F, smooth except at the marked points where it is C^1, and unique modulo weighted gauge transformations with the same regularity conditions, taking the weighted Higgs pair of the parabolic Higgs structure to a Yang–Mills–Higgs weighted pair.*

3. Orbifold and parabolic bundles

In this section we briefly discuss orbifold bundles, or 'V-bundles', in § 3 *a*, and the correspondence between holomorphic V-bundles and parabolic bundles with rational weights, in § 3 *b*. Although none of this material is original a detailed understanding of it is useful in the sequel.

(*a*) *V-bundles*

We start with some brief definitions (see Satake 1956; Scott 1983). By an *orbifold Riemann surface* we mean a closed, connected, complex 1-orbifold. We can think of an orbifold Riemann surface as a Riemann surface together with a finite number (assumed non-zero) of 'marked' points with, at each marked point, an associated order of isotropy n (an integer greater than one). (As before, we will usually assume that there is exactly one marked point.) Thus the Riemann surface M with marked point p becomes the 'underlying' surface of an orbifold Riemann surface once we specify an order of isotropy, n. Such a surface we write $\tilde{M}$.

Let σ denote the standard representation of $\mathbb{Z}_n$ on $\mathbb{C}$ as the nth roots of unity and D^2 the open unit disk in $\mathbb{C}$. Because $\tilde{M}$ is an orbifold we think of the *marked* point p as having a 'coordinate neighbourhood' U modelled on $D^2/\mathbb{Z}_n$. We can pull-back the n-fold branched covering $D^2 \twoheadrightarrow D^2/\mathbb{Z}_n$ and the action of $\mathbb{Z}_n$ to obtain

$\widehat{U} \longrightarrow \widehat{U}/\mathbb{Z}_n = U$; $\widehat{U}$ is called a *uniformizing coordinate neighbourhood*. The local holomorphic coordinate in $\widehat{U}$ is called a *uniformizing local coordinate*.

For convenience, we will work with a fixed choice of coordinate neighbourhood of p, $U \cong D^2/\mathbb{Z}_n$, and the corresponding uniformizing coordinate neighbourhood, $\widehat{U} \cong D^2$. The uniformizing local coordinate in $\widehat{U}$ will be denoted by z, with $r = |z|$ and $\theta = \arg(z)$, and the corresponding local coordinate in U by $w = z^n$. (Notice that $|z|^2 = |w|^{2/n}$ is smooth in the *orbifold* sense.)

Locally near a marked point a complex *vector V-bundle*, $E \to \tilde{M}$, with fibre $\mathbb{C}^l$ has a 'trivialization' $E|_U \stackrel{\simeq}{\to} (D^2 \times \mathbb{C}^l)/(\sigma \times \tau)$, where τ is an isotropy representation $\tau : \mathbb{Z}_n \to GL_l(\mathbb{C})$. Then we have by pull-back $\widehat{E}|_{\widehat{U}} \stackrel{\simeq}{\to} D^2 \times \mathbb{C}^l$ with a $\mathbb{Z}_n$-action given by $\sigma \times \tau$. Of course E is smooth (respectively holomorphic) if the transition functions are smooth (respectively holomorphic); the sections of E over U are defined to be the smooth (respectively holomorphic) $\mathbb{Z}_n$-equivariant sections of $\widehat{E}$ over $\widehat{U}$.

Similarly, any auxiliary structure which can be defined by a local definition and a patching condition (e.g. a metric or a connexion) has a V-bundle analogue: on $E|_U$ we define the structure $\mathbb{Z}_n$-equivariantly on $\widehat{E}|_{\widehat{U}}$. In particular, we will make much use of (orbifold) Riemannian metrics on $\tilde{M}$ and Hermitian metrics and connexions on vector V-bundles. The first Chern class or degree of a V-bundle can be defined using Chern–Weil theory; notice that the degree of a V-bundle is a *rational* number. Sobolev spaces and Hodge theory for V-bundles follow in the same way. (See § 4 *b* for comments on Sobolev spaces on orbifolds.)

About the marked point we can always choose a local trivialization of a complex vector V-bundle which *respects the V-structure*: that is, if the isotropy representation is $\tau : \mathbb{Z}_n \to GL_l(\mathbb{C})$ then we can choose coordinates so that τ decomposes as $\tau = \sigma^{x_1} \oplus \sigma^{x_2} \oplus \ldots \oplus \sigma^{x_l}$, where, for $i = 1, \ldots, l$, x_i is an integer with $0 \leqslant x_i < n$ and the x_i are increasing. (The requirement that the x_i are increasing means that these are what are called 'good' coordinates in Furuta & Steer (1992).) The x_i are called the *isotropy* of the V-bundle at the marked point.

(*b*) *Parabolic bundles and V-bundles*

The correspondence between holomorphic V-bundles and parabolic bundles with rational weights relies on a construction which is similar to the construction of a point bundle on a Riemann surface or an orbifold Riemann surface (Nasatyr & Steer 1995). Let $\mathcal{E}$ be a holomorphic V-bundle over $\tilde{M}$ with isotropy $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_l$. Let $\phi : \widehat{\mathcal{E}}|_{\widehat{U}} \stackrel{\simeq}{\to} D^2 \times \mathbb{C}^l$ be a fixed local holomorphic trivialization which respects the V-structure. Now define a holomorphic bundle $\mathcal{F} \to M$ by

$$\mathcal{F} = (\mathcal{E}|_{\tilde{M}\setminus\{p\}}) \cup_s D^2 \times \mathbb{C}^l,$$

with the clutching function s given on U by its $\mathbb{Z}_n$-equivariant lifting to $\widehat{U}$ which, by a small abuse of notation, we also denote by s, with

$$\left.\begin{array}{rcl} s\phi^{-1} : (D^2 \setminus \{0\}) \times \mathbb{C}^l & \to & D^2 \times \mathbb{C}^l \\ (z, (z_1, \ldots, z_l)) & \mapsto & (z^n, (z^{-x_1} z_1, \ldots, z^{-x_l} z_l)). \end{array}\right\} \tag{4}$$

Notice that the construction depends on the fixed choice of local uniformizing coordinate z and on the choice of *holomorphic* trivialization. (In the remainder of this paper we will also consider an analogous clutching construction involving weighted *unitary* frames.)

Now a holomorphic section of $(D^2 \times \mathbb{C}^l)/(\sigma \times \tau)$ is given by holomorphic maps $s_i : D^2 \to \mathbb{C}$, for $i = 1, \ldots, l$, equivariant with respect to the action of $\mathbb{Z}_n$. The equivariance condition and Taylor's theorem together imply that

$$s_i(z) = z^{x_i} s_i'(z^n), \tag{5}$$

where s_i' is a holomorphic function $s_i' : D^2 \to \mathbb{C}$. Under the map s defined by (4) we simply get a section of $(D^2 \setminus \{0\}) \times \mathbb{C}^l$ which is given by the functions $s_i'(w)$ and hence extends to a holomorphic section of $D^2 \times \mathbb{C}^l$. In other words the map s induces an isomorphism between the sheaves of germs of holomorphic sections.

In fact $\mathcal{F}$ has a natural parabolic structure as follows: working in a local trivialization which respects the V-structure we define weights $\lambda_1, \ldots, \lambda_l$ by $\lambda_i = x_i/n$. Define a flag in $F_p \cong \mathbb{C}^l$ according to the distinct isotropies of E, so that the smallest proper flag space is the eigenspace of $\mathbb{C}^l$ on which τ acts like σ^{x_l} and so on. The corresponding quasi-parabolic structure is then given by the image of this flag. With the weights λ_i it is clear that $\mathcal{F}$ is a parabolic bundle. The correspondence is reversible.

Theorem 3.1. (Furuta–Steer). *Let $\tilde{M}$ be an orbifold Riemann surface, with a single marked point p with order of isotropy n. Then construction above gives a bijection between holomorphic V-bundles over $\tilde{M}$ (modulo isomorphism) and parabolic bundles over M, parabolic at p with rational weights of the form x/n (modulo parabolic isomorphism). Moreover, there is an induced isomorphism of analytic sheaves between the sheaves of germs of holomorphic sections. The correspondence preserves subobjects, rank and degree and hence (semi-)stability.*

The theorem is more or less obvious from the construction except that we have to discuss the action of the two automorphism groups near the marked point. If g is an automorphism of E which makes two holomorphic structures isomorphic then locally we can take holomorphic trivializations which respect the V-structure and then g is holomorphic. Hence the corresponding map on F, sgs^{-1}, is also holomorphic (in the corresponding trivializations), except possibly at the marked point. However, if we consider the Taylor series for g, we find, analogous to (5),

$$g_{ij}(z) = \begin{cases} z^{x_i - x_j} g_{ij}'(z^n) & \text{if } x_i \geqslant x_j, \\ z^{n + x_i - x_j} g_{ij}'(z^n) & \text{if } x_i < x_j, \end{cases}$$

for holomorphic functions g_{ij}'. Conjugating by s we see that the result is not only holomorphic but the components with $x_i < x_j$ vanish at the marked point; in other words sgs^{-1} is a parabolic automorphism. A similar argument shows the converse.

From theorem 3.1 there is a satisfactory correspondence between (semi-)stable V-bundles and (semi-)stable parabolic bundles, provided the parabolic weights are *rational*. Suppose now that we have a weighted bundle $F \to M$ with a set of weights which are not necessarily rational. The arguments of Mehta & Seshadri (1980, § 2) show that a set of rational weights can be found arbitrarily close to the given weights and such that

(i) if $\mathcal{F}$ is any holomorphic structure on F then the parabolic bundles defined by $\mathcal{F}$ and the two sets of weights are (semi-)stable or not together and

(ii) if the parabolic structures defined by $\mathcal{F}$ and the two sets of weights are semistable then they have precisely the same destabilizing parabolic subbundles.

In many ways this is very satisfactory but we are not much nearer to actually constructing Yang–Mills (weighted unitary) connexions on parabolic bundles, which is the aim of this paper. Even given the correspondence between holomorphic V-bundles and parabolic bundles with rational weights discussed above it is not entirely straightforward to take a Yang–Mills connexion on a V-bundle and produce a corresponding Yang–Mills weighted unitary connexion on the parabolic bundle; we postpone tackling this point until § 5 *b*, where we give a discussion which is sufficiently general to apply also in the case of irrational weights (though it is not too difficult when one recalls that the connexions are flat up to twisting).

4. Connexions with singularities on V-bundles

In this section we will suppose that a Hermitian weighted bundle, possibly with irrational weights, is given and set up the Narasimhan–Seshadri problem on an appropriate V-bundle by allowing certain types of singular connexion. We introduce 'connexions with model singularity' in § 4 *a*, discuss their basic analytical properties in § 4 *b* and prove a version of Uhlenbeck's weak compactness theorem for them in § 4 *c*.

(*a*) *Connexions with model singularity*

If F is a Hermitian weighted bundle with (possibly irrational) weights $\lambda_1, \ldots, \lambda_l$ then we can construct a 'nearby' Hermitian V-bundle $\tilde{F}$, over an orbifold $\tilde{M}$ with underlying space M, by a unitary variant of the holomorphic clutching construction given in § 3 *b*. Of course this depends on choosing n, the order of isotropy for the marked point, and $x_1, \ldots, x_l$, the isotropy of the V-bundle—we will explain how these are chosen in § 4 *b*. Given the construction of $\tilde{F}$ we need to define a suitable space of connexions to work in—the space of 'connexions with model singularity.'

We fix a weighted unitary frame for F about the marked point and choose appropriate n and $x_1, \ldots, x_l$. Then the unitary clutching construction is to clutch to this fixed frame for F with the weighted unitary clutching map,

$$t = \begin{pmatrix} z^{-x_1} r^{x_1 - \lambda_1 n} & & 0 \\ & \ddots & \\ 0 & & z^{-x_l} r^{x_l - \lambda_l n} \end{pmatrix}. \tag{6}$$

(Compare (4).) Call the resulting V-bundle $\tilde{F}_{n;x_1,\ldots,x_l}$ or simply $\tilde{F}$. The point of this clutching construction is that the weighted Hermitian metric on F pulls-back to a *genuine* Hermitian metric on $\tilde{F}$, as is easily checked, and we will use this metric for our analysis. Of course, $\tilde{F}$ comes with a fixed unitary frame about the marked point which we will use for most of our local calculations. The construction depends on our fixed choices of local holomorphic coordinate, w, and weighted unitary frame.

Now suppose that in addition a holomorphic structure on F is given. We can consider a holomorphic frame about the marked point, related to the fixed weighted unitary frame by a weighted change of frame, g_0. The corresponding frame for $\tilde{F}$ (related to the fixed unitary frame by $\tilde{g}_0 = t^{-1} g_0 t$) will no longer be holomorphic because t, unlike s, isn't meromorphic. However, we can calculate the result of pulling the $\overline{\partial}$-operator back to $\tilde{F}$ via t (expressed with respect to this frame):

$$\overline{\partial} \mapsto \overline{\partial} + t^{-1}(\overline{\partial} t) = \overline{\partial} + \begin{pmatrix} x_1 - \lambda_1 n & & 0 \\ & \ddots & \\ 0 & & x_l - \lambda_l n \end{pmatrix} \frac{\mathrm{d}\overline{z}}{2\overline{z}}. \tag{7}$$

Applying the Chern construction we obtain a singular connexion which we term the *initial singular connexion* and denote A_0.

We sum up the situation in the following commutative diagram.

$$\begin{array}{ccc} \text{Unitary frame for } \tilde{F} & \xrightarrow{\ t\ } & \text{Weighted unitary frame for } F \\ \downarrow{\scriptstyle \tilde{g}_0} & & \downarrow{\scriptstyle g_0} \\ \begin{array}{c}\text{`Singular holomorphic' frame for } \tilde{F}, \\ \overline{\partial}\text{-operator is } \overline{\partial} + (x_i - \lambda_i n)\delta_{ij}\mathrm{d}\overline{z}/(2\overline{z})\end{array} & \xrightarrow[\ t\ ]{} & \text{Holomorphic frame for } F \end{array}$$

If it happens, improbably, that $g_0 = I$ so that the weighted unitary and holomorphic frames coincide, then the connexion associated to the $\overline{\partial}$-operator (7) by the Chern construction is just

$$d - \begin{pmatrix} x_1 - \lambda_1 n & & 0 \\ & \ddots & \\ 0 & & x_l - \lambda_l n \end{pmatrix} \mathrm{i}\mathrm{d}\theta.$$

Although usually $g_0 \neq I$ we will use this as the model singularity for our connexions on $\tilde{F}$.

For a real vector $\underline{\kappa} = (\kappa_1, \ldots, \kappa_l)$, with $\kappa_i = \kappa_j$ if $x_i = x_j$, let

$$\Lambda_{\underline{\kappa}} = \begin{pmatrix} \kappa_1 & & 0 \\ & \ddots & \\ 0 & & \kappa_l \end{pmatrix} \mathrm{i}\mathrm{d}\theta. \tag{8}$$

Now fix $\epsilon_i = x_i - \lambda_i n$ and set $\Lambda = \Lambda_{\underline{\epsilon}}$ with $(1,0)$- and $(0,1)$-parts $\Lambda^{1,0}$ and $\Lambda^{0,1}$, respectively. Let B_0 be any connexion which is smooth and unitary over the complement of the marked point and near the marked point has the form $d - \Lambda$ (in the unitary frame fixed by the clutching construction). Call B_0 the *model connexion.* A *connexion with model singularity* is then any unitary connexion which differs from B_0 by something smooth. (It is clear that the definition doesn't depend on the particular choice of B_0.) We will often refer simply to a 'singular connexion'. (Since singularities other than the model singularity will occur from § 4 *b* on, we will sometimes use the oxymoronic term 'smooth singular connexion' to emphasize that the connexion is smooth *modulo the model singularity.*)

We have defined connexions with the particular singularity that arises out of our clutching construction (given by $\epsilon_i = x_i - \lambda_i n$): equally, for a fixed local uniformizing coordinate z and any Hermitian V-bundle with a fixed unitary frame around the marked point, we can define *connexions with $\underline{\kappa}$-singularity* or '$\underline{\kappa}$-singular connexions' for any $\underline{\kappa}$.

For any $\underline{\kappa}$ we set

$$\mu(\underline{\kappa}) = \frac{1}{l}\left\{\mathrm{Deg}\,(\tilde{F}) - \frac{1}{n}\sum_{i=1}^{l}\kappa_i\right\}$$

the 'slope' of $\underline{\kappa}$-singular connexions; notice that $\mu(\underline{\epsilon}) = \mathrm{WeiDeg}\,(F)/l$ is the parabolic slope. Given a Riemannian metric, whether smooth on M, or orbifold on $\tilde{M}$, we say that a $\underline{\kappa}$-singular connexion A is *Yang–Mills* (with respect to the given metric) if $F_A = -2\pi\mathrm{i}\mu(\underline{\kappa})(*1)I$, where '$*$' denotes the Hodge star of the metric.

Remark 1. For $\underline{\kappa}$-singular connexions the Chern–Weil formula

$$\int_{\tilde{M}} \mathrm{tr}\,(F_A) = -2\pi\mathrm{i}l\mu(\underline{\kappa})$$

is easily proved (Biquard 1991, proposition 2.9). One can also see that any $\underline{\kappa}$-singular connexion has a well-defined limit holonomy around the marked point, depending only on $\underline{\kappa}$.

Remark 2. Notice that the initial singular connexion A_0 (the pull-back of the Chern connexion of a holomorphic structure on F) usually is not a connexion with model singularity because it differs from B_0 by something which is not smooth. This difference will, however, have a large number of derivatives (see corollary 4.6) and so will lie in a suitable Sobolev space.

(*b*) *Analytical properties of singular connexions*

We use suitable Sobolev spaces of 'L^2_1 singular connexions,' both in order to have all the usual machinery for nonlinear analysis available and because the initial connexion given by a holomorphic structure is not contained in the space of (smooth) singular connexions. (We always work on a Hermitian V-bundle over a *compact* base.) The analytical estimates of proposition 4.2 and proposition 4.3 show that the singularity in our connexions is counteracted by vanishing at the marked point, which is implied by equivariance conditions. We start, however, by explaining how to choose n and the x_is so that the unitary clutching construction of § 4 *a* has sufficiently good analytical properties. We could also work with L^p_1 singular connexions, for $p > 1$, but the estimates in proposition 4.3, which are essential in the sequel (particularly in the proof of proposition 4.9), follow from the fact that L^2_k is a Hilbert space. Therefore, with the exception of the more general proposition 4.2 and in § 6, we work with $p = 2$, contenting ourselves with a few remarks concerning other values of p.

Let $F \to M$ be a Hermitian weighted bundle. For a given integer $k_0 \geqslant 2$, we say that $\tilde{F} = \tilde{F}_{n;x_1,\ldots,x_l}$ (the V-bundle over $\tilde{M}$ constructed by the unitary clutching construction in § 4 *a*) is a k_0*-approximation to* F if n and $x_1, \ldots, x_l$ have the following properties:

(i) n satisfies

$$\frac{k_0 - 1}{n} < |\lambda_i - \lambda_j| < 1 - \frac{k_0 - 1}{n} \quad \text{for all } \lambda_i \neq \lambda_j \tag{9}$$

and if any of the λ_is are rational then they can be expressed with denominator n;

(ii) the x_i/ns have the same pattern of equalities and inequalities as the λ_is and

$$|\lambda_i - x_i/n| < 1/2n \quad \text{for all } i \tag{10}$$

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(i.e. the x_i/ns are the best possible rational approximations to the λ_is with denominator n).

Clearly for any given $k_0 \geqslant 2$ a k_0-approximation to F exists simply by taking n large enough and choosing the x_is appropriately. Notice that (9) and (10) together imply that

$$k_0 - 2 < |x_i - x_j| < n - k_0 + 2 \quad \text{for all } x_i \neq x_j. \tag{11}$$

A k_0-approximation to F will then have sufficiently good analytical properties provided k_0 is sufficiently large (e.g. see lemma 4.1 or proposition 4.9).

Suppose from now on that $\tilde{F}$ is a k_0-approximation to a Hermitian weighted bundle F for some k_0. In fact we will suppose that $k_0 \geqslant 4$ and other, less explicit, bounds on k_0 will also be introduced in the course of this paper.

Lemma 4.1. *If g is a weighted automorphism of F and t is as in (6) then $\tilde{g} = t^{-1}gt$ is an $L^2_{k_0}$ automorphism of the V-bundle $\tilde{F}$.*

Proof. For $\tilde{g}_{ij}(z) = g_{ij}(z^n) z^{x_i - x_j} r^{n(\lambda_i - \lambda_j) - (x_i - x_j)}$, which clearly satisfies the equivariance condition and so defines an automorphism of $\tilde{F}$. Moreover $\tilde{g}_{ij}$ has order at least $n(\lambda_i - \lambda_j)$ in r: in fact if $\lambda_j > \lambda_i$ then, because g is a weighted automorphism, g_{ij} must vanish to at least first order in $|w| = r^n$, so that the order is actually $n(1 + \lambda_i - \lambda_j)$. If $\lambda_i = \lambda_j$ then we just have g_{ij}, which is smooth. Otherwise, under the condition (9), the order in r is strictly greater than $k_0 - 1$ and the result follows. ∎

Now let $\mathcal{A}^2_\Lambda$ be the space of L^2_1 *connexions with model singularity* or 'L^2_1 singular connexions' on $\tilde{F}$, i.e. the space of unitary connexions on $\tilde{F}$ which differ from the base connexion B_0 by something in L^2_1. Let $\mathcal{G}^2$ be the group of L^2_2 V-bundle gauge transformations (i.e. L^2_2 automorphisms of $\tilde{F}$, fixing the base and preserving the metric) and $(\mathcal{G}^c)^2$ its complexification (i.e. L^2_2 automorphisms of $\tilde{F}$, fixing the base). Notice that lemma 4.1 shows that $\tilde{g}$ is in $(\mathcal{G}^c)^2$, as we assume $k_0 \geqslant 4$.

We should emphasize that we are using the standard gauge group for the Hermitian V-bundle $\tilde{F}$, which doesn't depend on the parabolic weights (or, more precisely, depends on them only to the extent that they determine the choice of $\tilde{F}$).

We now give some estimates concerning Sobolev spaces of sections of V-bundles. Sections of a V-bundle are given locally by sections which are equivariant in a local uniformizing coordinate: Sobolev spaces of sections can then be defined locally by completing the smooth equivariant sections with respect to a given Sobolev norm. The equivariance condition means that the smooth equivariant sections of a line V-bundle with isotropy x will have vanishing $(\min\{x, n-x\} - 1)$-jet. (In fact, provided $k \leqslant \min\{x, n-x\}$, it is easy to see that one gets exactly the same L^p_k Sobolev space by completing the smooth equivariant sections which are *supported away from the marked point.*)

The crucial fact about such L^p_k spaces of sections is that under suitable conditions the map $f \mapsto f/r$ is bounded from $L^p_k \to L^p_{k-1}$. This allows us to to deal with the model singularity (as $|\mathrm{d}\theta| = 1/r$) and, for instance, to show that $\mathcal{A}^2_\Lambda$ is closed under the natural action of $(\mathcal{G}^c)^2$ and that the initial singular connexion A_0 (given by the holomorphic structure on F) is in $\mathcal{A}^2_\Lambda$ (see corollaries 4.4, 4.5 and 4.6). The required estimates are given in the following two propositions. Proposition 4.2 deals with general p and is included only for completeness; it can be passed over without any loss. (The proof is based on the comments of the referees and also owes something to lemma 3.5 of Kronheimer & Mrowka (1993); we originally gave a proof along the lines of Biquard (1991, théorème 1.3), which we found to be less revealing.) Proposition 4.3 gives simpler proofs of the estimates and necessary additional results, for the particular case $p = 2$.

Proposition 4.2. *For $p > 1$ and $k \geqslant 1$, suppose that f is the limit of an L^p_k-Cauchy sequence of smooth $\mathbb{C}$-valued functions supported within the unit disc, each obeying an equivariance condition with isotropy x such that $\min\{x, n-x\} \geqslant k$ (so that the $(k-1)$-jet vanishes at the origin). Then for non-negative integers i and j with $i + j \leqslant k$*

$$\left\| \frac{f}{r^i} \right\|_{L^p_j} \leqslant c(i,j)\, \|f\|_{L^p_{i+j}}$$

where $c(i,j)$ is a positive constant. Analogous estimates for division by powers of z and $\bar{z}$ also hold.

Proof. The case $i = j = 0$ is trivial. We consider first the case $i = 1$, $j = 0$. Let C_r be the circle centre the origin of any radius $0 < r \leqslant 1$. The equivariance condition implies that f satisfies $\int_{Cr} f \, \mathrm{d}\theta = 0$. Simple integration by parts shows that, on C_r, f is given by a convolution:

$$f = -\frac{1}{2\pi} \frac{\partial f}{\partial \theta} * \theta.$$

(Here and in the rest of the proof, Fubini's theorem shows that the integral exists for almost all $r \in (0, 1]$.) It follows that

$$\begin{aligned} \left| \frac{f}{r} \right| &\leqslant \int_{C_r} \left| \frac{1}{r} \frac{\partial f}{\partial \theta} \right| \mathrm{d}\theta \\ &\leqslant (2\pi)^{1-1/p} \left(\int_{C_r} \left| \frac{1}{r} \frac{\partial f}{\partial \theta} \right|^p \mathrm{d}\theta \right)^{1/p} \end{aligned}$$

for $p \geqslant 1$, using Hölder's inequality. Hence

$$\begin{aligned} \left\| \frac{f}{r} \right\|_{L^p} &= \left(\int_{r=0}^{1} \int_{C_r} \left| \frac{f}{r} \right|^p \mathrm{d}\theta \, r \, \mathrm{d}r \right)^{1/p} \\ &\leqslant 2\pi \left(\int_{r=0}^{1} \int_{C_r} \left| \frac{1}{r} \frac{\partial f}{\partial \theta} \right|^p \mathrm{d}\theta \, r \, \mathrm{d}r \right)^{1/p} \\ &= 2\pi \left\| \frac{1}{r} \frac{\partial f}{\partial \theta} \right\|_{L^p}. \end{aligned}$$

Finally, comparing ∇f to $(1/r)\partial f / \partial \theta$, the result for $i = 1$, $j = 0$ follows. Iterating the above argument shows that

$$\left\| \frac{f}{r^i} \right\|_{L^p} \leqslant (2\pi)^i \left\| \frac{1}{r^i} \frac{\partial^i f}{\partial \theta^i} \right\|_{L^p},$$

which gives the result for any i and $j = 0$.

Now assume inductively that the result holds up to and including $j - 1$. Applying the inductive hypothesis (and suppressing constants) we estimate

$$\begin{aligned}\left\|\frac{f}{r^i}\right\|_{L^p_j} &\leqslant \left\|\frac{f}{r^i}\right\|_{L^p} + \left\|\frac{\nabla f}{r^i}\right\|_{L^p_{j-1}} + \left\|\frac{f}{r^{i+1}}\right\|_{L^p_{j-1}} \\ &\leqslant \|f\|_{L^p_i} + \|\nabla f\|_{L^p_{j+i-1}} + \|f\|_{L^p_{j+i}} \\ &\leqslant c(i,j)\,\|f\|_{L^p_{j+i}}\end{aligned}$$

as required. ■

Proposition 4.3. *For $k \geqslant 1$, suppose that f is the limit of an L^2_k-Cauchy sequence of smooth $\mathbb{C}$-valued functions supported within the unit disc, each obeying an equivariance condition with isotropy x such that $\min\{x, n-x\} \geqslant k$ (so that the $(k-1)$-jet vanishes at the origin). Then*

(i) for non-negative integers i and j with $i + j \leqslant k$

$$\left\|\frac{f}{r^i}\right\|_{L^2_j} \leqslant d(i,j)\,\|f\|_{L^2_{i+j}},$$

where $d(i,j)$ is a positive constant;

(ii) $d(1,0) \leqslant 1/\min\{x, n-x\}$ and

(iii) the map $f \mapsto f/r^i$ from equivariant L^2_{i+j} functions to equivariant L^2_j functions is compact.

Analogous results for division by powers of z and $\bar{z}$ also hold.

Proof. We consider the case $i = 1$, $j = 0$ first. For almost all fixed r, f is L^2 and hence equal to its Fourier series; $f = \sum_{-\infty<m<\infty} \hat{f}_m(r)\mathrm{e}^{\mathrm{i}m\theta}$. By the equivariance condition $\hat{f}_m = 0$ for $|m| < \min\{x, n-x\}$ and so we can estimate

$$\begin{aligned}\left\|\frac{f}{r}\right\|_{L^2} &= \sum_{-\infty<m<\infty} \left\|\frac{\hat{f}_m(r)}{r}\mathrm{e}^{\mathrm{i}m\theta}\right\|_{L^2} \\ &\leqslant \frac{1}{\min_{\hat{f}_m\neq 0}\{|m|\}} \sum_{-\infty<m<\infty} |m| \left\|\frac{\hat{f}_m(r)}{r}\mathrm{e}^{\mathrm{i}m\theta}\right\|_{L^2} \\ &\leqslant \frac{1}{\min\{x, n-x\}} \left\|\frac{1}{r}\frac{\partial f}{\partial \theta}\right\|_{L^2}.\end{aligned}$$

Comparing ∇f to $(1/r)\partial f/\partial\theta$, we obtain the estimate for $i = 1$, $j = 0$, including the claim that $d(1,0) \leqslant 1/\min\{x, n-x\}$. The estimate for any i and j follows exactly as in proposition 4.2.

We will prove compactness in the case $i = 1$, $j = 0$, after which the general case follows immediately. So suppose that $\{f_l\}_{l\in\mathbb{N}}$ is a sequence of functions as in the statement of the proposition, with a universal bound $\|f_l\|_{L^2_1} \leqslant M$, for all $l \in \mathbb{N}$. Considering $\|(1/r)\partial f/\partial\theta\|_{L^2}$ we see that $\{\|m\hat{f}^{(l)}_m \mathrm{e}^{\mathrm{i}m\theta}/r\|_{L^2}\}_{l\in\mathbb{N}}$ is bounded for each fixed m and so a subsequence converges. Hence by a standard diagonal argument we can assume that, after passing to a subsequence and relabelling, $\{\|m\hat{f}^{(l)}_m \mathrm{e}^{\mathrm{i}m\theta}/r\|_{L^2}\}_{l\in\mathbb{N}}$ converges for all m.

Now consider

$$\left\| \frac{f_l - f_{l'}}{r} \right\|_{L^2} \leqslant \frac{1}{\min\{x, n-x\}} \sum_{|m| \leqslant N} \left\| m \frac{\hat{f}_m^{(l)} - \hat{f}_m^{(l')}}{r} \mathrm{e}^{\mathrm{i}m\theta} \right\|_{L^2} + \frac{1}{N} \sum_{|m| > N} \left\| m \frac{\hat{f}_m^{(l)} - \hat{f}_m^{(l')}}{r} \mathrm{e}^{\mathrm{i}m\theta} \right\|_{L^2},$$

for any $N \in \mathbb{N}$. The second term on the right-hand side is no greater than $2M/N$ (using the bound and the triangle inequality) and so can be made arbitrarily small by taking N sufficiently large. The first term on the right-hand side is a finite sum of norm-differences in Cauchy sequences and so can be made arbitrarily small by taking l and l' sufficiently large. Hence the sequence $\{f_l/r\}_{l \in \mathbb{N}}$ is Cauchy in L^2. ■

Remark 1. In both cases we could have been more explicit about the constants involved. However, the case $p = 2$, $i = 1$, $j = 0$ is the only one for which an explicit estimate will be required in the sequel, as well as being the easiest.

Remark 2. In each case a little juggling allows us to deal with division by non-integral powers of r: for any y with $-k \leqslant y < 1$ and any non-negative integer j with $[y] \leqslant j \leqslant k + [y]$

$$\|r^y f\|_{L^p_j} \leqslant c'([y], j) \, \|f\|_{L^p_{j-[y]}},$$

where $c'(p, [y], j)$ is a positive constant. To see this we simply calculate, suppressing constants,

$$\begin{aligned} \|r^y f\|_{L^p_j} &\leqslant \sum_{a+b \leqslant j} \left\| r^{y-a} \nabla^b f \right\|_{L^p} \\ &\leqslant \sum_{a+b \leqslant j} \left\| r^{[y]-a} \nabla^b f \right\|_{L^p} \left\| r^{y-[y]} \right\|_{L^\infty} \\ &\leqslant \sum_{a+b \leqslant k+[y]} \|f\|_{L^p_{a+b-[y]}}, \end{aligned}$$

using proposition 4.2 or proposition 4.3 for the last line.

Corollary 4.4. *Let g be an L^2_k endomorphism of the V-bundle $\tilde{F}$ and a an L^2_{k-1} endomorphism-valued 1-form both supported within the D^2-neighbourhood of the marked point. Let $\Lambda = \Lambda_{\underline{\epsilon}}$, as in (8), with $\epsilon_i = x_i - \lambda_i n$. Then the operators*

$$\begin{aligned} L^2_k &\to L^2_{k-1} \\ g &\mapsto [g, \Lambda] \end{aligned}$$

and

$$\begin{aligned} L^2_{k-1} &\to L^2_{k-2} \\ a &\mapsto [a, \Lambda] \end{aligned}$$

are bounded linear maps provided $k \leqslant k_0 - 1$. In the second case, if $k = 2$ the operator norm is no greater than $1/(k_0 - 2)$. These results hold equally if Λ is replaced by $\Lambda^{1,0}$ or $\Lambda^{0,1}$.

Proof. First note that

$$[g, \Lambda]_{ij} = g_{ij}(\epsilon_i - \epsilon_j)\mathrm{i}\mathrm{d}\theta.$$

Since $|\mathrm{d}\theta| = 1/r$ the strategy of the proof is to use the estimate for the L^2_{k-1}-norm of $g_{ij}(\epsilon_i - \epsilon_j)/r$ in terms of the L^2_k-norm of $g_{ij}(\epsilon_i - \epsilon_j)$ provided by proposition 4.3 (and similarly for a).

The isotropy for the matrix entry g_{ij} is $x_i - x_j$. Similarly, writing $a = a^{1,0} + a^{0,1}$, for a_{ij} we have that $a^{1,0}_{ij}$ has isotropy $x_i - x_j - 1$ and $a^{0,1}_{ij}$ has isotropy $x_i - x_j + 1$.

If $\epsilon_i = \epsilon_j$ then the operator is identically zero. If not, then $x_i \neq x_j$. Then, by (11), for g_{ij} we have $\min\{x, n - x\} \geqslant k_0 - 1$ and for a_{ij} we have $\min\{x, n - x\} \geqslant k_0 - 2$. Also note that $|\epsilon_i - \epsilon_j| < 1$ by (10). Hence, applying proposition 4.3 with $j = k - 1$ in the first case and $j = k - 2$ in the second and $i = 1$, we obtain the result including the estimate of the operator norm when $k = 2$. ■

The important point about the estimate of the operator norm of $a \mapsto [a, \Lambda]$ is that it can be made arbitrarily small by taking k_0 large enough; this will be essential for the proofs of propositions 4.9 and 4.10. The case $p = 2$ is the only one in which an estimate depending on k_0 in this way is immediate—the proof of proposition 4.3 relied on the fact that L^2 is a Hilbert space—but the interpolation argument of the Riesz–Thorin convexity theorem (see Ahlfors 1966) shows that, as a function of p, the operator norm can be supposed continuous at $p = 2$.

Corollary 4.5. *The group $(\mathcal{G}^{\mathrm{c}})^2$ of L^2_2 automorphisms of $\tilde{F}$ acts on $\mathcal{A}^2_\Lambda$ and connexions in $\mathcal{A}^2_\Lambda$ have L^2 curvature.*

Proof. Consider first the action of $g \in (\mathcal{G}^{\mathrm{c}})^2$ on $A \in \mathcal{A}^2_\Lambda$. Using the Sobolev multiplication lemma we see that it is sufficient to consider $g(B_0) - B_0$; that this is L^2_1 follows immediately from corollary 4.4. The proof that the curvature is L^2 is similar. ■

Corollary 4.6. *If A_0 is the initial singular connexion on $\tilde{F}$ determined by a holomorphic structure on F then $A_0 \in \mathcal{A}^2_\Lambda$, i.e. A_0 is an L^2_1 singular connexion. Moreover, there exists $\tilde{g}_0 \in (\mathcal{G}^{\mathrm{c}})^2$ such that $\tilde{g}_0(A_0)$ is given by $d - \Lambda$ with respect to the fixed unitary frame about the marked point.*

Proof. The only problem here is near the marked point. We work in the unitary frame fixed by the clutching construction; since we are in a unitary frame it is sufficient to check only the $(0,1)$-part of the connexion. This frame will be mapped to that corresponding to the holomorphic frame on F by $\tilde{g}_0 = t^{-1}g_0 t$ (where g_0 the weighted change of frame on F from the weighted unitary frame to the holomorphic frame), which is L^2_2 by lemma 4.1. Applying (7) we see that the pulled-back $\overline{\partial}$-operator is given by $\overline{\partial} + \tilde{g}_0^{-1}(\overline{\partial}\tilde{g}_0) - \tilde{g}_0^{-1}\Lambda^{0,1}\tilde{g}_0$ and the second claim is clear. For the first claim we need to show that

$$\tilde{g}_0^{-1}(\overline{\partial}\tilde{g}_0) - \tilde{g}_0^{-1}\Lambda^{0,1}\tilde{g}_0 + \Lambda^{0,1} = \tilde{g}_0^{-1}(\overline{\partial}\tilde{g}_0) - [\tilde{g}_0^{-1}, \Lambda^{0,1}]\tilde{g}_0$$

is L^2_1; this follows from corollary 4.4 (and the Sobolev multiplication lemma). ■

Remark. Because we will only be interested in the $(\mathcal{G}^{\mathrm{c}})^2$-orbit of A_0 we can apply the second part of the corollary and always suppose that A_0 has the form $d - \Lambda$ with respect to the fixed unitary frame. This justifies our definition of the model connexion B_0 and with A_0 in this form we can also suppose that $B_0 = A_0$.

(c) *Weak compactness for singular connexions*

Now we give a weak compactness theorem for $\mathcal{A}^2_\Lambda$, following Uhlenbeck (1982). As the proof is well known we merely sketch it, drawing attention to the points where allowances need to be made for the model singularity. The main addition to Uhlenbeck's proof is the use of corollary 4.4 to show that $\|[a,\Lambda]\|_{L^2}$ is controlled by $\|a\|_{L^2_1}/(k_0-2)$. We concentrate on the proof of the local result (Uhlenbeck 1982, theorem 1.3) in a neighbourhood of the marked point; the global theorem will then follow in the usual way, provided k_0 is sufficiently large. Since we only intend to apply the global theorem to sequences of Yang–Mills connexions we only give a version for sequences such that the L^2 curvature doesn't concentrate at points, which is easier to derive.

The proof of the local result is in rough outline as follows. We want to prove that in a D^2-neighbourhood of the marked point all connexions which are in $\mathcal{A}^2_\Lambda$ and have sufficiently small curvature in the L^2-norm have a certain property. The proof is to show that the set of all connexions with a given bound on the L^2-norm of the curvature is connected and that the subset of those with the desired property is non-empty, closed and open, if the bound is small enough. We want to adapt lemmas 2.5–2.8 of Uhlenbeck (1982): 2.6 and 2.8 generalize immediately and 2.5 and 2.7 exhibit the same subtlety. We have the following version of 2.5 (compare also Biquard (1991, lemme 1.9)).

Lemma 4.7. *There exist constants k and k' (not depending on k_0) such that if a is an L^2_1 skew-Hermitian matrix-valued 1-form over the closed unit disc $\overline{D^2}$ (defined by the local uniformizing coordinate z) with*

$$d^*a = 0 \quad \text{and}$$
$$a_r = 0 \quad \text{on } \partial\overline{D^2},$$

then

$$\|a\|_{L^2_1} \leqslant 4k\,\|F_{d-\Lambda+a}\|_{L^2}$$

provided $\|\Lambda\| \leqslant 1/2k$ and $\|a\|_{L^4} \leqslant 1/4kk'$. (Here a_r denotes the radial component of a and $\|\Lambda\|$ denotes the operator norm of the map $a \mapsto [a,\Lambda]$, from L^2_1 to L^2, of corollary 4.4.)

Proof. Since the boundary condition is elliptic for $d\oplus d^*$ and there is no kernel we get an elliptic inequality

$$\|a\|_{L^2_1} \leqslant k\,\|da\|_{L^2}. \tag{12}$$

Calculating the curvature we find

$$\begin{aligned}\|F_{d-\Lambda+a}\|_{L^2} &\geqslant \|da\|_{L^2} - \|[a,\Lambda]\|_{L^2} - \|a\wedge a\|_{L^2}\\ &\geqslant (1/k)\,\|a\|_{L^2_1} - \|\Lambda\|\,\|a\|_{L^2_1} - \|a\|^2_{L^4}\\ &\geqslant \{(1/k) - \|\Lambda\| - k'\,\|a\|_{L^4}\}\,\|a\|_{L^2_1}.\end{aligned}$$

Here we have used (12), corollary 4.4 and, for the last line, the Sobolev inequality.

It follows that, provided $\|\Lambda\| \leqslant 1/2k$, we can obtain the desired estimate: for instance, if $\|a\|_{L^4} \leqslant 1/4kk'$ then

$$\|a\|_{L^2_1} \leqslant 4k\,\|F_{d-\Lambda+a}\|_{L^2},$$

as required. (The constants arise from elliptic and Sobolev inequalities valid for all functions, not just equivariant ones, and so are independent of k_0.) ■

The generalization of lemma 2.7 of Uhlenbeck (1982) follows in exactly the same way, with the same condition on $\|\Lambda\|$ and the same constant $1/4kk'$. By corollary 4.4, we know that we can suppose $\|\Lambda\|$ is sufficiently small if $k_0 \geqslant 2k+2$ and so a local result follows.

Proposition 4.8. *Let k be the constant of lemma 4.7 and suppose that $k_0 \geqslant \max\{2k+2, 4\}$. Then there exists a constant $c > 0$ such that if a' is an L^2_1 skew-Hermitian matrix-valued 1-form over the closed unit disc $\overline{D^2}$ (defined by the local uniformizing coordinate z) with $\|F_{d-\Lambda+a'}\|_{L^2} < c$ then there exists a gauge transformation taking $d - \Lambda + a'$ to $d - \Lambda + a$ with*

$$d^* a = 0,$$
$$a_r = 0 \quad \text{on } \partial\overline{D^2} \quad \text{and}$$
$$\|a\|_{L^2_1} \leqslant 4k \, \|F_{d-\Lambda+a}\|_{L^2}.$$

(Here a_r denotes the radial component of a.)

Our global result like theorem 3.6 of Uhlenbeck (1982) follows in the standard way.

Proposition 4.9. *Suppose that F is a Hermitian weighted bundle and $\tilde{F}$ a k_0-approximation to F with $k_0 \geqslant \max\{2k+2, 4\}$, where k is the constant of lemma 4.7. Suppose further that $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of connexions in $\mathcal{A}^2_\Lambda(\tilde{F})$ with the property that for any $x \in \tilde{M}$ there exists a geodesic ball D_x such that $\left\|F_{A_j}\chi_{D_x}\right\|_{L^2} < c$ for all sufficiently large j, where c is the constant of proposition 4.8. Then $\{A_j\}_{j\in\mathbb{N}}$ has a subsequence which is weakly convergent in $\mathcal{A}^2_\Lambda$ modulo L^2_2 changes of gauge.*

Remark 1. We have stated the result with reference to the Hermitian weighted bundle F. Of course, it is really a result about V-bundles; one simply needs to replace the fact that $\tilde{F}$ is a k_0-approximation to F by an estimate on the isotropy analogous to (11).

Remark 2. The Yang–Mills equation coupled with radial gauge-fixing and a suitable equivariance condition determines a connexion matrix completely in a neighbourhood of the marked point (see § 5 *b*). Hence a version of proposition 4.9 for sequences of Yang–Mills connexions can be proved using radial gauge-fixing around the marked point, rather than the Coulomb gauge-fixing used in proposition 4.8, to obtain L^2_1-bounds on connexion matrices.

Remark 3. In order to prove the stronger result that weak convergence holds given only a uniform L^2 bound on the curvatures of the sequence one needs also to work with the L^p_1 norm for some p with $4/3 < p < 2$. The only problem then is to control the norm of the map $a \mapsto [a, \Lambda]$ from L^p_1 to L^p but this can be done if p is sufficiently close to 2 by the comments following corollary 4.4.

We close this section by considering a larger space of connexions in which the model singularity may vary. Let $\tilde{F}$ be a k_0-approximation to F with $k_0 \geqslant \max\{2k+2, 4\}$, as in proposition 4.9. Fix a smooth bump function $\psi(r)$, supported within the D^2-neighbourhood of the marked point. Now the space of connexions which are L^2_1 with

$\underline{\kappa}$-singularity for some $\underline{\kappa} \in \mathbb{R}^l$ is just $\mathcal{A}^2_\Lambda \times \mathbb{R}^l$, with

$$(A, \underline{\kappa}) \leftrightarrow A + \psi \Lambda_{\underline{\kappa}-\underline{\epsilon}}, \tag{13}$$

where $\Lambda_{\underline{\kappa}-\underline{\epsilon}}$ is defined by (8). Of course, we give $\mathcal{A}^2_\Lambda \times \mathbb{R}^l$ the product topology and we have the obvious action of the gauge group on the right-hand side. (Strictly, there may be less than l independent parameters for the types of singularity as $\kappa_i = \kappa_j$ if $x_i = x_j$: we can avoid any additional notation by supposing that, for the remainder of this section only, l denotes the number of *unequal* x_is.)

Now we consider weak compactness for $\mathcal{A}^2_\Lambda \times \mathbb{R}^l$. Whenever $\|\Lambda_{\underline{\kappa}}\| \leqslant 1/2k$ a version of proposition 4.9 follows for connexions with $\underline{\kappa}$-singularity. Moreover, if K is a *compact* neighbourhood of $\underline{\epsilon} \in \mathbb{R}^l$ (compactness of K ensures weak compactness of $L^2_k(D^2) \times K$) such that this inequality holds for each $\underline{\kappa} \in K$ then the result holds even when $\underline{\kappa}$ is allowed to vary in K. In fact $\|\Lambda_{\underline{\kappa}}\| \leqslant 2\max_i |\kappa_i| / (k_0 - 2)$ (for the estimate in corollary 4.4 we simply used (10) to conclude that $|\epsilon_i| < 1/2$) so that K can be taken to consist of all $\underline{\kappa}$s with $|\kappa_i| \leqslant 1/2$ for all i, i.e. $K = [-1/2, 1/2]^l$. Thus we have the following more general result.

Proposition 4.10. *Suppose that F is a Hermitian weighted bundle and $\tilde{F}$ a k_0-approximation to F with $k_0 \geqslant \max\{2k+2, 4\}$, where k is the constant of lemma 4.7. Suppose further that $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of connexions on $\tilde{F}$ corresponding, via (13), with a sequence in $\mathcal{A}^2_\Lambda \times [-1/2, 1/2]^l$ and with the property that for any $x \in \tilde{M}$ there exists a geodesic ball D_x such that $\|F_{A_j}\chi_{D_x}\|_{L^2} < c$ for all sufficiently large j, where c is the constant of proposition 4.8. Then $\{A_j\}_{j\in\mathbb{N}}$ has a subsequence which is weakly convergent in $\mathcal{A}^2_\Lambda \times [-\frac{1}{2}, \frac{1}{2}]^l$ modulo L^2_2 changes of gauge.*

5. The proof of the Narasimhan–Seshadri theorem

In this section we finally prove a version of the Narasimhan–Seshadri theorem for stable parabolic bundles and weighted unitary connexions. The section starts with some analytical preliminaries and remarks on the $\overline{\partial}$-operator associated to a singular connexion on an orbifold $\tilde{M}$. Next, § 5 *b* is concerned with comparing orbifold Riemannian metrics on $\tilde{M}$ and smooth Riemannian metrics on M and with local descriptions of Yang–Mills connexions about the marked point: these results will be used to pass from a Yang–Mills singular connexion on $\tilde{F}$ to a Yang–Mills weighted unitary connexion on F, where $\tilde{F}$ and F are as in § 4. The core of the proof is in § 5 *c*, where we solve the problem for irrational weights on the parabolic bundle by choosing an approximating sequence of rational weights and solving the problem for each set of rational weights in turn. Thus we get a sequence of Yang–Mills L^2_1 singular connexions with varying model singularities corresponding to the varying weights on a k_0-approximation to the Hermitian weighted bundle: if $k_0 \geqslant \max\{2k+2, 4\}$ then we can apply proposition 4.10 to get the result.

(*a*) *Singular $\overline{\partial}$-operators*

Any connexion $A \in \mathcal{A}^2_\Lambda$ has an associated $\overline{\partial}$-operator, $\overline{\partial}_A$, which we call an L^2_1 *$\overline{\partial}$-operator with model singularity* or 'L^2_1 singular $\overline{\partial}$-operator'. These operators, being modeled on $\overline{\partial} - \Lambda^{0,1}$ and not being smooth, do not define holomorphic structures on $\tilde{F}$ in any obvious way. Since the proof of the theorem requires algebraic-geometric information about such operators we need to associate holomorphic structures (on F or $\tilde{F}$) to them. Compare the following proposition with Biquard (1992).

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Proposition 5.1. *Each* $(\mathcal{G}^c)^2$*-orbit in* $\mathcal{A}^2_\Lambda$ *contains a singular connexion which takes the form*

$$d - \Lambda$$

in the fixed unitary frame about the marked point.

Proof. We work with the singular $\overline{\partial}$-operator and adapt the proof of the Newlander–Nirenberg theorem given in Atiyah & Bott (1982, § 5): first we solve the problem globally on a particularly simple orbifold Riemann surface and then a patching argument gives a general, local solution.

The problem is to find an L^2_2 automorphism g such that, in a neighbourhood of the marked point and with respect to our fixed frame,

$$\overline{\partial} + g^{-1}(\overline{\partial} g) - g^{-1}\Lambda^{0,1}g + g^{-1}a^{0,1}g = \overline{\partial} - \Lambda^{0,1},$$

for some given L^2_1 endomorphism-valued $(0,1)$-form, $a^{0,1}$. At least locally, we can define a mapping $\Theta : g \mapsto \{\overline{\partial} g + [g, \Lambda^{0,1}]\}g^{-1}$; then the problem is simply to solve $\Theta(g) = a^{0,1}$ (for equivariant g and $a^{0,1}$).

Let $\tilde{M}$ be the orbifold Riemann surface which is $\mathbb{CP}^1$ with 0 and ∞ marked with order of isotropy n. Let $\widehat{L} = \mathbb{CP}^1 \times \mathbb{C}$ be the trivial complex line-bundle over $\mathbb{CP}^1$ and define an action of $\mathbb{Z}_n$ by

$$\mathrm{e}^{2\pi \mathrm{i}/n}(z, z') = (\mathrm{e}^{2\pi \mathrm{i}/n} z, \mathrm{e}^{2x\pi \mathrm{i}/n} z'), \tag{14}$$

for some integer x. Then the quotient $L_x = \widehat{L}/\mathbb{Z}_n$ is a V-bundle over $\tilde{M}$. The isotropy of L_x is x at 0 and $-x$ at ∞ and sections of L_x are sections of $\widehat{L}$ (i.e. functions on $\mathbb{CP}^1$) equivariant with respect to the $\mathbb{Z}_n$-action given by (14). We note that the V-bundle of $(0,1)$-forms on $\tilde{M}$ is just L_1.

Now we set $E = \oplus_{i=1}^{l} L_{x_i}$, which, at 0, is locally like $\tilde{F}$ at the marked point, and try to solve the problem globally on E. If a solution exists, for $\|a^{0,1}\|_{L^2_1}$ sufficiently small, then the result follows using a cut-off function, exactly as in Atiyah & Bott (1982, § 5) (although the choice of the cut-off function requires some care). By the implicit function theorem it suffices to show that the derivative at the identity (on the L^2_2 endomorphisms of E), $D\Theta_I$, is surjective. Of course E is far from trivial but its lift to $\mathbb{CP}^1$, $\widehat{E}$, is and we can consider Θ as being defined on the equivariant automorphisms of $\widehat{E}$, globally over $\mathbb{CP}^1$. On $\widehat{E}$ we have $D\Theta_I : h \mapsto \overline{\partial} h + [h, \Lambda^{0,1}]$. We calculate

$$\{D\Theta_I(h)\}_{ij} = \overline{\partial} h_{ij} + \frac{(\epsilon_i - \epsilon_j)h_{ij}}{2\overline{z}}\,\mathrm{d}\overline{z}.$$

It is easy to see that the equivariance conditions mean that the image is an End (E)-valued $(0,1)$-form on $\tilde{M}$, as expected.

So it suffices to show that for each fixed pair (i,j) the operator $\partial/\partial\overline{z} + (\epsilon_i - \epsilon_j)/(2\overline{z})$ surjects from the L^2_2 equivariant functions representing sections of $L_{x_j - x_i}$ onto the L^2_1 equivariant functions representing $L_{x_j - x_i}$-valued $(0,1)$-forms. For $\epsilon = \epsilon_i - \epsilon_j$ consider the operator

$$\widehat{D}_\epsilon = \frac{\partial}{\partial \overline{z}} + \frac{\epsilon}{2\overline{z}}, \tag{15}$$

on functions over $\mathbb{CP}^1$. We consider $\widehat{D}_\epsilon$ restricted to the L^2_2 completion of the equivariant functions with equivariance condition specified by $x = x_j - x_i$ and denote the resulting operator by $D_{\epsilon,x}$. (Notice that $x = 0$ only if $\epsilon = 0$.) It follows from corollary 4.4 that $D_{\epsilon,x}$ defines a bounded linear operator from L_2^2 equivariant functions to L_1^2 equivariant functions.

If $\epsilon = 0$ then $D_{0,x} = \bar{\partial}$. Now $\bar{\partial}$ is Fredholm and its index is given by the orbifold Riemann–Roch theorem (Kawasaki 1979); it is 1 if $x = 0$ and 0 otherwise. The kernel of $\bar{\partial}$ over $\mathbb{CP}^1$ consists of rational functions but these can only be L_2^2 everywhere if they are constant. However, the constant functions on $\mathbb{CP}^1$ don't descend to $\tilde{M}$ unless $x = 0$ and so $D_{0,x}$ has 1-dimensional kernel if $x = 0$ and is injective otherwise. It follows that $D_{0,x}$ is surjective.

Now suppose that $\epsilon \neq 0$ (and so $x \neq 0$). By proposition 4.3 $D_{\epsilon,x} - \bar{\partial}$ is compact and so $D_{\epsilon,x}$ is Fredholm and of the same index as $\bar{\partial}$. As above, the orbifold Riemann–Roch theorem gives the index of $\bar{\partial}$ on L_x as 0. Therefore $D_{\epsilon,x}$ is surjective provided it is injective.

We consider the problem over $\mathbb{CP}^1$: if $s \in \ker \widehat{D}_\epsilon$ then we substitute $s' = sr^\epsilon$ and find that $\bar{\partial}s' = 0$ and so s' is rational. As before, s being L_2^2 forces s' to be 0. So $D_{\epsilon,x}$ is injective if $\epsilon \neq 0$ and hence surjective. This concludes the proof. ∎

Remark. The last paragraph of the proof brings out the basic fact about zeros of the local singular $\bar{\partial}$-operator (15): if $s \in \ker \widehat{D}_\epsilon$, smooth except possibly at the marked point, then $s = s'r^{-\epsilon}$, where s' is holomorphic except possibly at the marked point, where it is meromorphic.

Corollary 5.2. *Each $(\mathcal{G}^{\mathrm{c}})^2$-orbit of L_1^2 singular connexions on $\tilde{F}$ contains a smooth singular connexion.*

Proof. For this proof we adapt lemmas 14.6–14.8 of Atiyah & Bott (1982). However, to avoid problems with the model singularity we consider a singular connexion in the form given by proposition 5.1 over some neighbourhood of the marked point and consider only the subgroup of $(\mathcal{G}^{\mathrm{c}})^2$ consisting of automorphisms which are the identity over this neighbourhood. In this restricted context we get no problems with the model singularity and lemmas 14.6–14.8 generalize easily. ∎

There is an alternative proof of proposition 5.1 as a corollary of proposition 5.4, below.

We *do not* claim an analogue of lemma 14.9 of Atiyah & Bott (1982) for singular connexions: indeed, one would not expect such a result given that the clutching construction and hence $\Lambda^{0,1}$ are not smooth. However, we certainly can bootstrap up to a certain point using corollary 4.4 and conclude that if g is L_2^2 and intertwines two smooth singular $\bar{\partial}$-operators, then g is smooth away from the marked point and L_k^2 there, provided k_0 is sufficiently large (we need $k_0 \geqslant k+1$ and sufficiently large with respect to certain elliptic constants $c_3, \ldots, c_k$).

Since the model singularity arises from the singular clutching map t of (6), we *do* get a version of lemma 14.9 of Atiyah & Bott (1982) by translating back to the weighted bundle, F, as follows. If we have two singular $\bar{\partial}$-operators which are not necessarily smooth but come from smooth objects on F, i.e. from two holomorphic structures on F, tgt^{-1} is smooth away from the marked point and, modulo smooth weighted changes of frame, is meromorphic at the marked point. However, as in the proof of proposition 5.1, because g is L_2^2 in z, $\{tgt^{-1}\}_{ij}$ must have a removeable singularity at the marked point and even a zero there if $\lambda_i < \lambda_j$. In other words tgt^{-1} is a (smooth) weighted automorphism.

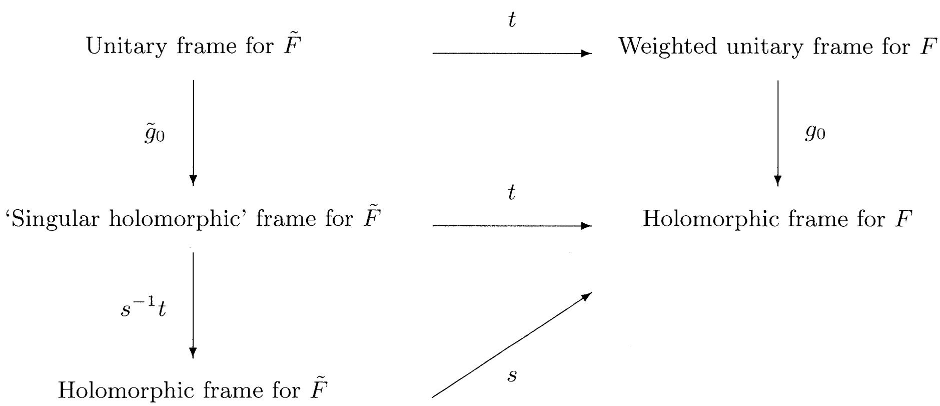
Consider now a smooth singular connexion on $\tilde{F}$ locally given in the form of proposition 5.1. The associated singular $\overline{\partial}$-operator has the form $\overline{\partial} - \Lambda^{0,1}$ locally and clearly the result of clutching by t is to produce a smooth $\overline{\partial}$-operator on F locally like $\overline{\partial}$, i.e. a holomorphic structure such that the fixed trivialization is holomorphic. Now this defines a bijection between the L_1^2 singular $\overline{\partial}$-operators on $\tilde{F}$ modulo L_2^2 automorphisms and holomorphic structures on F modulo weighted automorphisms; to see this, a discussion of the germs of automorphisms, as previously outlined for the proof of theorem 3.1, is necessary.

This last result can also be seen without constructing 'good' gauges by comparing the holomorphic and unitary clutching constructions as follows. Suppose that F is a Hermitian weighted bundle with a distinguished holomorphic structure and use a holomorphic trivialization to define the holomorphic clutching construction (using the clutching map s of (4)) (note that the holomorphic clutching construction depends only on the flag on F and not the weights themselves and that both constructions certainly produce the same smooth V-bundle, $\tilde{F}$). We obtain a distinguished holomorphic structure on $\tilde{F}$ or *initial* $\overline{\partial}$*-operator*, denoted $\overline{\partial}_0$. So we have two clutching constructions of $\tilde{F}$ given by

$$s = \begin{pmatrix} z^{-x_1} & & 0 \\ & \ddots & \\ 0 & & z^{-x_l} \end{pmatrix} \tag{16}$$

and t and related by $s^{-1}t$. Considering $s^{-1}t$ as an automorphism of $\tilde{F}$ we find that its effect on L_1^p singular $\overline{\partial}$-operators is to remove the model singularity. The resulting $\overline{\partial}$-operator on $\tilde{F}$ may not be smooth but we will show that its orbit under a suitable space of (Sobolev) automorphisms contains a smooth point.

Again we sum up the relation between the holomorphic and unitary clutching constructions in a commutative diagram.



Recall that $\epsilon_i = x_i - \lambda_i n$. Now, by (10), there exists $0 < \delta < 1$ such that

$$|\epsilon_i| < \tfrac{1}{2}(1-\delta) \quad \text{for all } i. \tag{17}$$

Choose and fix q with

$$2 < q < \frac{2}{1-\delta} \tag{18}$$

(the reason for this condition will become apparent in the proof of lemma 5.3). For q fixed as above, we want to show that the result of acting on an L_1^2 singular $\overline{\partial}$-operator by $s^{-1}g_0t$ is an L^q $\overline{\partial}$-operator.

Lemma 5.3. *If q satisfies (18) then, on $\tilde{F}$, $s^{-1}g_0t$ takes L_1^2 singular $\overline{\partial}$-operators to L^q $\overline{\partial}$-operators and L_2^2 automorphisms to L_1^q automorphisms. Under this correspondence the $\overline{\partial}$-operator of the initial singular connexion A_0 is mapped to the initial $\overline{\partial}$-operator $\overline{\partial}_0$.*

Proof. We factorize $s^{-1}g_0t = (s^{-1}t).\tilde{g}_0$. By lemma 4.1 and corollary 4.5 $\tilde{g}_0$ acts on the L_1^2 singular connexions and L_2^2 automorphisms. It remains to consider the action of

$$s^{-1}t = \begin{pmatrix} r^{\epsilon_1} & & 0 \\ & \ddots & \\ 0 & & r^{\epsilon_l} \end{pmatrix}$$

around the marked point. Suppose that $a^{0,1}$ is a local L_1^2 endomorphism-valued $(0,1)$-form such that in the 'singular holomorphic' frame our singular $\overline{\partial}$-operator is given by $\overline{\partial} - \Lambda^{0,1} + a^{0,1}$. The action of $s^{-1}t$ is then

$$\begin{aligned} \overline{\partial} - \Lambda^{0,1} + a^{0,1} &\mapsto \overline{\partial} + s^{-1}t\overline{\partial}(t^{-1}s) - s^{-1}t\Lambda^{0,1}t^{-1}s + s^{-1}ta^{0,1}t^{-1}s \\ &= \overline{\partial} + s^{-1}ta^{0,1}t^{-1}s. \end{aligned}$$

The claim that A_0 and $\overline{\partial}_0$ correspond is now clear.

To see that $s^{-1}ta^{0,1}t^{-1}s$ is L^q we note that, using (17) and (18), $r^{\epsilon_i-\epsilon_j}$ is $L^{q+\iota}$ for suitably small $\iota > 0$. Since $a^{0,1}$ is L_1^2 it is certainly $L^{q(q+\iota)/\iota}$. Then by Hölder's inequality $\|a_{ij}^{0,1}r^{\epsilon_i-\epsilon_j}\|_{L^q}$ is finite.

Similarly, consider $\{s^{-1}tgt^{-1}s\}_{ij} = g_{ij}r^{\epsilon_i-\epsilon_j}$. If $\epsilon_i = \epsilon_j$ then there is no problem. On the other hand, if $\epsilon_i \neq \epsilon_j$, we can write

$$\|g_{ij}r^{\epsilon_i-\epsilon_j}\|_{L_1^q} \leqslant \|g_{ij}r^{\epsilon_i-\epsilon_j}\|_{L^q} + \|\nabla(g_{ij})r^{\epsilon_i-\epsilon_j}\|_{L^q} + \left\|\frac{g_{ij}}{r}r^{\epsilon_i-\epsilon_j}\right\|_{L^q}.$$

Now, since g_{ij}, ∇g_{ij} and g_{ij}/r are all L_1^2 (the latter by proposition 4.3), we can apply the argument given above for $a^{0,1}$ to obtain the desired result. Note that, because $q > 2$, the L_1^q automorphisms form a Lie group acting smoothly on the L^q $\overline{\partial}$-operators (compare corollary 4.5). ■

Proposition 5.4. *A $(\mathcal{G}^c)^2$-orbit of L_1^2 singular connexions on $\tilde{F}$ determines a holomorphic structure on $\tilde{F}$, unique up to the action of $\mathcal{G}^c$. Conversely, a holomorphic structure on $\tilde{F}$ determines an L_1^2 singular connexion, unique up to the action of $(\mathcal{G}^c)^2$.*

Proof. By lemma 5.3 the data determines an orbit of L^q $\overline{\partial}$-operators (with q as in (18)) under the action of the L_1^q automorphisms and we claim that each such orbit contains a smooth point, unique up to smooth isomorphism, as required. To prove the claim we simply note that lemmas 14.6–14.9 of Atiyah & Bott (1982) apply immediately (the crucial point is that $1 - 2/q > 0$, i.e. $q > 2$, so that the Sobolev multiplication lemma holds).

The proof of the converse is similar to the proof of lemma 5.3: we consider a smooth $\bar{\partial}$-operator which is locally $\bar{\partial} + b^{0,1}$ and a smooth automorphism g and estimate $\|b^{0,1}_{ij} r^{\epsilon_j - \epsilon_i}\|_{L^2_1}$ and $\|g_{ij} r^{\epsilon_j - \epsilon_i}\|_{L^2_2}$. As in lemma 5.3, the case $\epsilon_i = \epsilon_j$ presents no problem so we suppose that $\epsilon_i \neq \epsilon_j$. Then g_{ij}/r^2 is L^2_2 by proposition 4.3, $r^{2+\epsilon_j-\epsilon_i}$ is also L^2_2 and the product is L^2_2 by the Sobolev multiplication lemma. Similarly, $b^{0,1}_{ij}/r^2$ and $r^{1+\epsilon_j-\epsilon_i}$ and their product are all L^q_1 (we avoid L^2_1 as it is borderline for the Sobolev multiplication lemma). ∎

Remark. As mentioned above, proposition 5.4 leads to a simple proof of proposition 5.1 from the corresponding fact for smooth $\bar{\partial}$-operators on V-bundles, i.e. the Newlander–Nirenberg theorem.

By virtue of proposition 5.4 we can pass from (unitary) L^2_1 singular connexions on the Hermitian copy of $\tilde{F}$ constucted via the unitary clutching construction to holomorphic structures on the copy of $\tilde{F}$ constucted via the holomorphic clutching construction. The latter correspond to parabolic structures on F by theorem 3.1. (Strictly, theorem 3.1 applies when F is given rational weights $x_1/n, \ldots, x_r/n$ but of course parabolic isomorphism classes of parabolic structures depend only on the holomorphic and quasi-parabolic structures and not the weights themselves.)

We may sum up by noting that, as a corollary of proposition 5.1 (or proposition 5.4) we have bijections

$$\frac{\mathcal{A}^2_\Lambda(\tilde{F})}{(\mathcal{G}^{\mathrm{c}})^2(\tilde{F})} \leftrightarrow \frac{\mathcal{A}(\tilde{F})}{\mathcal{G}^{\mathrm{c}}(\tilde{F})} \leftrightarrow \frac{\mathcal{A}(F)}{\mathcal{G}^{\mathrm{c}}_{\mathrm{wei}}(F)}$$

and, by corollary 5.2, the natural map

$$\mathcal{A}_\Lambda(\tilde{F}) \to \frac{\mathcal{A}^2_\Lambda(\tilde{F})}{(\mathcal{G}^{\mathrm{c}})^2(\tilde{F})}$$

is surjective. (Here, of course, $\mathcal{A}_\Lambda(\tilde{F})$ denotes singular $\bar{\partial}$-operators, $\mathcal{A}(\tilde{F})$ and $\mathcal{A}(F)$ denote smooth $\bar{\partial}$-operators or holomorphic structures and $\mathcal{G}^{\mathrm{c}}(\tilde{F})$ denotes the group of smooth automorphisms—$\mathcal{A}^2_\Lambda(\tilde{F})$, $(\mathcal{G}^{\mathrm{c}})^2(\tilde{F})$ and $\mathcal{G}^{\mathrm{c}}_{\mathrm{wei}}(F)$ are as before.)

(*b*) *Local forms of Yang–Mills connexions*

Given a Hermitian weighted bundle F with a stable parabolic structure we will produce a Yang–Mills singular connexion on the V-bundle $\tilde{F}$ and we would like to carry this solution back to F and give suitable regularity and uniqueness results there. The required results follow from the fact that in a *radial gauge* the vanishing that follows from equivariance ensures that a Yang–Mills singular connexion has a very particular form; see proposition 5.6 and compare lemma A.1 of Jeffrey (1992). (This argument is certainly particular to two real dimensions.)

Our first problem is that the Yang–Mills condition on $\tilde{F}$ is defined with respect to an *orbifold* Riemannian metric. In order to adapt our result to the parabolic bundle we would first like a version which uses a *smooth* Riemannian metric. Near the marked point we have a uniformizing coordinate neighbourhood $\widehat{U}$ with uniformizing coordinate z and a corresponding coordinate neighbourhood U with local coordinate $w = z^n$. Suppose that we have an orbifold Riemannian metric and a smooth Riemannian metric, both normalized to unit volume. Denote the resulting Hodge stars by

$*_z$ and $*_w$ respectively. The V-bundles of 0- and 2-forms both have trivial isotropy and so the ordinary 0- and 2-forms can also be considered as orbifold forms. Hence we can compare the actions of $*_z$ and $*_w$ on 2-forms. For example, if we suppose that locally the metrics are $\mathrm{d}z \otimes \mathrm{d}\overline{z}$ and $\mathrm{d}w \otimes \mathrm{d}\overline{w}$ then an easy calculation shows that for a local 2-form ω

$$*_z\,\omega = n^2 r^{2(n-1)}(*_w\omega) \qquad \text{and} \qquad *_z\,1 = n^{-2}r^{-2(n-1)}(*_w 1). \tag{19}$$

Given a Yang–Mills singular connexion on $\tilde{F}$ we would like to prove that a singular connexion which is Yang–Mills with respect to the *smooth* metric exists on $\tilde{F}$. (Here we can work with model singularity $\Lambda_{\underline{\kappa}}$ for any $\underline{\kappa}$.) To do this we simply use orbifold Hodge theory to construct an appropriate automorphism of the V-bundle which will take a singular connexion which is Yang–Mills with respect to the orbifold Riemannian metric into the required form. (This is much like the rank-1 case of the Narasimhan–Seshadri theorem.)

Proposition 5.5. *If A is a $\underline{\kappa}$-singular connexion on $\tilde{F}$ which is Yang–Mills with respect to the orbifold Riemannian metric then there exists a smooth positive real-valued function $g : \tilde{M} \to \mathbb{R}^+$, such that $(gI)(A)$ is Yang–Mills with respect to the smooth Riemannian metric.*

Proof. Recall that both metrics are assumed normalized to unit volume. By hypothesis we have that

$$F_A = -2\pi \mathrm{i}\mu(\underline{\kappa})(*_z 1)I$$

and we are required to construct a connexion with curvature

$$-2\pi \mathrm{i}\mu(\underline{\kappa})(*_w 1)I.$$

Set

$$\omega = -2\pi \mathrm{i}\mu(\underline{\kappa})\{(*_w 1) - (*_z 1)\}$$

in orbifold deRham cohomology. Note that ω is cohomologous to zero and hence exact. The result will follow if we can construct a positive real-valued g with $d((gI)(A) - A) = \omega I$, so that $F_{(gI)(A)} = -2\pi \mathrm{i}\mu(\underline{\kappa})(*_w 1)I$. For gI (a particularly simple type of bundle automorphism) we have

$$\begin{aligned}(gI)(A) - A &= (g^{-1}\partial g + g\overline{\partial} g^{-1})I \\ &= (\partial - \overline{\partial})(\ln g)I,\end{aligned}$$

so that we need to solve

$$\begin{aligned}\omega &= (\overline{\partial}\partial - \partial\overline{\partial})(\ln g) \\ &= 2\overline{\partial}\partial(\ln g).\end{aligned} \tag{20}$$

Now we use a little (orbifold) Hodge theory. Let G be the Green's operator of the orbifold Laplacian, Δ. Since ω is exact we have that $\omega = \Delta G\omega$. Now, on 2-forms the Laplacian is just $2\mathrm{i}\overline{\partial}\partial *_z$. It follows that a solution to (20) is given by setting

$$g = \exp\{\mathrm{i} *_z G\omega\}$$

and the proposition is proved. ∎

Suppose that A is an L^2_1 $\underline{\kappa}$-singular connexion on $\tilde{F}$, Yang–Mills with respect to the *smooth* Riemannian metric. In fact there is always a gauge in which A is smooth except at the marked point. This follows in the standard way by our local gauge-fixing result proposition 4.8 and a bootstrapping argument. By our remarks on regularity in § 5 *a* it is clear that a limited bootstrapping argument can be applied at the marked point: for instance we can certainly choose k_0 so that the singularity there is, at worst, L_2^2 and hence continuous (modulo the model singularity).

Now we place A in radial gauge. To this end we work locally in polar coordinates: although these are not well adapted to working with Sobolev spaces and elliptic estimates they are natural for discussing equivariance conditions and the vanishing at the marked point that these imply. Write A in the local trivialization as

$$\begin{aligned} A &= d - \Lambda_{\underline{\kappa}} + a^{1,0}\,\mathrm{d}z + a^{0,1}\,\mathrm{d}\bar{z} \\ &= d - \Lambda_{\underline{\kappa}} + a_r\,\mathrm{d}r + a_\theta\,\mathrm{d}\theta, \end{aligned}$$

where
$$a_r = \mathrm{e}^{\mathrm{i}\theta}a^{1,0} + \mathrm{e}^{-\mathrm{i}\theta}a^{0,1}, \quad a_\theta = \mathrm{i}(za^{1,0} - \bar{z}a^{0,1}).$$

We search for a change of gauge which will eliminate the radial component of the connexion matrix. This means finding a gauge transformation g such that $\partial g/\partial r + a_r g = 0$. For fixed θ, we consider this equation along each line segment $\{t\mathrm{e}^{\mathrm{i}\theta} : t \in (-1,1)\}$: we obtain an ordinary differential equation for $g(t)$ with continuous coefficients (both $a^{1,0}$ and $a^{0,1}$ are continuous and, by equivariance, vanish at the origin; therefore a_r is also continuous and vanishes at the origin). Hence, given the initial condition $g(0) = I$, there is a unique solution for each θ which also satisfies the equation $\partial(g^*g)/\partial t = 0$ and so is unitary. For each θ, the solution is continuously differentiable in t at the origin and, since the same is true for any transverse line, we see that g is at least C^1 there (and smooth elsewhere). Moreover the solution is clearly equivariant. Of course, our hope is that $g \in \mathcal{G}^2$, i.e. g is L_2^2; we will eventually see that this is the case.

Now, the result of acting on A by g is a connexion which is locally

$$g(A) = d - \Lambda_{\underline{\kappa}} + a'_\theta\,\mathrm{d}\theta,$$

where
$$a'_\theta\,\mathrm{d}\theta = g^{-1}[g, \Lambda_{\underline{\kappa}}] + \left(g^{-1}\frac{\partial g}{\partial\theta} + g^{-1}a_\theta g\right)\mathrm{d}\theta,$$

i.e. $g(A)$ is in radial gauge. The most that we claim *a priori* for this connexion is that a'_θ is continuous and vanishes at the origin: the contribution of the term $g^{-1}[g, \Lambda_{\underline{\kappa}}]$ to a'_θ is actually C^1 (the singularity of $\Lambda_{\underline{\kappa}}$ is in the $\mathrm{d}\theta$) and the contribution of the other terms is continuous (as A is); the equivariance condition ensures that a'_θ vanishes at the origin.

Since g is unitary $g(A)$ is still Yang–Mills (with respect to the smooth Riemannian metric): let us consider the Yang–Mills equations. We suppose, without loss of generality, that the metric is locally like $\mathrm{d}w \otimes \mathrm{d}\bar{w} = n^2 r^{2(n-1)}\,\mathrm{d}z \otimes \mathrm{d}\bar{z}$ near the marked point. Therefore, in our radial gauge the Yang–Mills equations are just

$$\frac{1}{r}\frac{\partial a'_\theta}{\partial r} = -2\pi\mathrm{i}\mu(\underline{\kappa})n^2 r^{2(n-1)} I.$$

With the initial condition $a'_\theta(0) = 0$ this has the unique solution

$$a'_\theta = -\pi\mathrm{i}\mu(\underline{\kappa})(n/2)r^{2n} I,$$

locally. (Note that without the initial condition, which comes from equivariance, we could not determine a'_θ completely: for one consequence of this see the second remark following proposition 4.9.) Now

$$a'_\theta \, \mathrm{d}\theta = -\tfrac{1}{2}\pi \mathrm{i}\mu(\underline{\kappa})nr^{2n}I \, \mathrm{d}\theta = -\tfrac{1}{4}\pi\mu(\underline{\kappa})nr^{2(n-1)}I(\overline{z}\,\mathrm{d}z - z\,\mathrm{d}\overline{z})$$
$$= -\tfrac{1}{4}\pi\mu(\underline{\kappa})I(\overline{w}\,\mathrm{d}w - w\,\mathrm{d}\overline{w})$$

and so $g(A)$ is a smooth singular connexion.

Finally we consider regularity of g: since g is L^2_1 and maps between an L^2_2 singular connexion and a smooth one the bootstrap argument shows that it is certainly L^2_2. Combining the above discussion with proposition 5.5 we have the following result.

Proposition 5.6. *Let A be an L^2_1 $\underline{\kappa}$-singular connexion which is Yang–Mills with respect to a given orbifold Riemannian metric on $\tilde{M}$. Then there is a gauge transformation $g \in \mathcal{G}^2$ such that $g(A)$ is a smooth $\underline{\kappa}$-singular connexion, Yang–Mills with respect to a given smooth Riemannian metric on M and locally of the form,*

$$g(A) = d - \Lambda_{\underline{\kappa}} - \tfrac{1}{4}\pi\mu(\underline{\kappa})nr^{2(n-1)}I(\overline{z}\,\mathrm{d}z - z\,\mathrm{d}\overline{z}),$$

in our fixed unitary trivialization (if the metric is supposed to be like $\mathrm{d}w \otimes \mathrm{d}\overline{w}$ near the marked point).

(c) *The proof of the theorem*

In this subsection we at last prove our main theorem. As a first step we prove the following theorem on $\tilde{F}$; a further brief discussion then leads to the main result, theorem 2.1.

Theorem 5.7. *Let F be a Hermitian weighted bundle and $\tilde{F} = \tilde{F}_{n;x_1,\ldots,x_r}$ a k_0-approximation to F. Let $\mathcal{F}$ be a stable parabolic structure on F and let A_0 be the corresponding initial L^2_1 singular connexion on $\tilde{F}$. Then, provided k_0 is sufficiently large, there exists $g \in (\mathcal{G}^{\mathrm{c}})^2(\tilde{F})$ such that $g(A_0)$ is a smooth singular connexion, Yang–Mills with respect to a given smooth Riemannian metric and locally of the form*

$$g(A) = d - \Lambda - \tfrac{1}{4}\pi\mu(F)nr^{2(n-1)}I(\overline{z}\,\mathrm{d}z - z\,\mathrm{d}\overline{z}),$$

in our fixed unitary trivialization (if the metric is supposed to be like $\mathrm{d}w \otimes \mathrm{d}\overline{w}$ near the marked point). Moreover, g is unique up to the action of $\mathcal{G}^2(\tilde{F})$.

Our proof is based on the intuitive idea that a solution to the Narasimhan–Seshadri problem for a parabolic bundle with irrational weights can be obtained as the limit of solutions for the same bundle with rational weights; these solutions are obtained from the Narasimhan–Seshadri theorem for V-bundles. (A proof closely following Donaldson (1983) is also possible.)

As before, let F be a Hermitian weighted bundle with irrational weights, equipped with a stable parabolic structure, $\mathcal{F}$. Now choose a sequence of rational approximations $x_i^{(j)}/n_j$, $j = 1, 2, \ldots$, to the weights λ_i such that

(i) for each j the $x_i^{(j)}/n_j$s have the same pattern of equalities and inequalities as the λ_is;

(ii) $\tilde{F}_1 = \tilde{F}_{n_1;x_1^{(1)},\ldots,x_l^{(1)}}$ is a k_0-approximation to F with k_0 large enough that proposition 4.10 applies;

(iii) n_j divides n_{j+1} (strictly) for all $j \geqslant 1$;

(iv) $|(x_i^{(j)}/n_j) - \lambda_i| < 1/2n_j$ for all $j \geqslant 1$;

(v) $\mathcal{F}$ remains parabolically stable when the λ_is are replaced by the $x_i^{(j)}/n_j$s, for all $j \geqslant 1$.

(The last condition can certainly be satisfied by the argument of Mehta & Seshadri (1980) discussed in § 3 *b*.) The third and fourth conditions together show that $x_i^{(j)}/n_j \to \lambda_i$ as $j \to \infty$ for all $i = 1, \ldots, l$. For $i = 1, \ldots, l$, set $\epsilon_i^{(j)} = x_i^{(j)} - \lambda_i n_j$. We write $n = n_1$, $x_i = x_i^{(1)}$ for $i = 1, \ldots, l$ and $\tilde{F}$ for $\tilde{F}_1$: we will be doing our analysis on $\tilde{F}$ (over an orbifold $\tilde{M} = \tilde{M}_1$). Notice that on $\tilde{F}$ we have the initial singular connexion A_0 of § 4.

Define a singular automorphism of F, h_j, by setting

$$h_j = \begin{pmatrix} |w|^{\epsilon_1^{(j)}/n_j} & & 0 \\ & \ddots & \\ 0 & & |w|^{\epsilon_l^{(j)}/n_j} \end{pmatrix} \tag{21}$$

in the weighted unitary frame near the marked point, with h_j smooth elsewhere. Then if we act on the weighted Hermitian metric by h_j we obtain a weighted Hermitian metric with respect to the weights $x_i^{(j)}/n_j$: denote the resulting Hermitian weighted bundle with these weights by F_j.

Of course F_j still has a stable parabolic structure defined by $\mathcal{F}$ and the weights. Let $\tilde{F}_j \to \tilde{M}_j$ be the Hermitian V-bundle obtained from F_j via the weighted unitary clutching construction with clutching map (analogous to (6))

$$t_j = \begin{pmatrix} z_j^{-x_1^{(j)}} & & 0 \\ & \ddots & \\ 0 & & z_j^{-x_l^{(j)}} \end{pmatrix}, \tag{22}$$

where z_j is the local uniformizing coordinate on $\tilde{M}_j$ with order of isotropy n_j. (Since F_j has rational weights the unitary clutching map (22) and the holomorphic clutching map (16) are the same; the two constructions differ only by the frame chosen to clutch to.) Notice that t_j is meromorphic and so $\bar{\partial} t_j = 0$: applying theorem 3.1 and the argument of lemma 4.1 we obtain an L_1^2 initial connexion on $\tilde{F}_j$ (with *no* model singularity) in the $(\mathcal{G}^{\mathrm{c}})^2(\tilde{F}_j)$-orbit of a stable holomorphic structure. The Narasimhan–Seshadri theorem for V-bundles now gives us a Yang–Mills connexion in the same orbit. By proposition 5.6 we can suppose this connexion is smooth, Yang–Mills with respect to a given smooth Riemannian metric on M (supposed to be like $\mathrm{d}w \otimes \mathrm{d}\overline{w}$ near the marked point) and of the form

$$d - \tfrac{1}{4}\pi\mu(F_j) n_j r_j^{2(n_j-1)} I(\overline{z}_j \, \mathrm{d}z_j - z_j \, \mathrm{d}\overline{z}_j), \tag{23}$$

locally in our fixed unitary trivialization.

Recall that F_j has the same holomorphic structure as F but that the weighted Hermitian metric is altered by applying the singular automorphism h_j, defined by (21). Again we can summarise the relations between the various frames in a commutative

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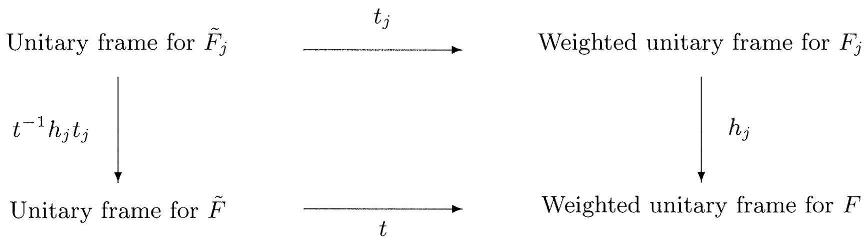
diagram.



By construction, the weighted unitary frames of F and F_j coincide, of course, and, away from the marked point, h_j is a unitary map between them. The result is that, again away from the marked point, $t^{-1}h_jt_j$ is a unitary map from $\tilde{F}_j$ to $\tilde{F}$.

Since $z = z_j^{n_j/n}$, we have $\{t^{-1}h_jt_j\}_{ik} = \delta_{ik}(z_j/r_j)^{-x_i^{(j)}+x_in_j/n}$ and the action of $t^{-1}h_jt_j$ on a connexion of the form (23) on $\tilde{F}_j$ gives us a (unitary) singular connexion A_j on $\tilde{F}$ of the form,

$$A_j = d - A_{(\underline{x}-\underline{x}^{(j)}n/n_j)} - \tfrac{1}{4}\pi\mu(F_j)nr^{2(n-1)}I(\bar{z}\mathrm{d}z - z\mathrm{d}\bar{z}),$$

where $\underline{x} = (x_1, \ldots, x_l)$ and $\underline{x}^{(j)} = (x_1^{(j)}, \ldots, x_l^{(j)})$. Thus we obtain a Yang–Mills connexion A_j with $(\underline{x}-\underline{x}^{(j)}n/n_j)$-singularity on $\tilde{F}$. Since $x_i^{(j)}/n_j \to \lambda_i$ and $\mu(F_j) \to \mu(F)$ as $j \to \infty$ curvature cannot concentrate at points and we can apply proposition 4.10 to the sequence $\{A_j\}$ to conclude that, modulo $\mathcal{G}^2(\tilde{F})$, there is a subsequence with a weak limit A_∞. Moreover, A_∞ is clearly Yang–Mills and has $\underline{\epsilon}$-singularity.

It remains to show that A_∞ is in the same orbit as the initial singular connexion A_0. To do this we apply the arguments of § 5 *a*: the Yang–Mills connexions obtained on the F_j all represent the same (stable) orbit in $\mathcal{A}(F)/\mathcal{G}^{\mathrm{c}}_{\mathrm{wei}}(F)$ (the weighted Hermitian metric varies with j but not the quasi-parabolic structure). Passing back to the V-bundle $\tilde{F}$ using the holomorphic clutching map s, we obtain a sequence of smooth $\bar{\partial}$-operators which locally have the form,

$$\bar{\partial}_j = \bar{\partial} + \tfrac{1}{4}\pi\mu(F_j)nr^{2(n-1)}Iz\,\mathrm{d}\bar{z} \qquad \text{and} \qquad \bar{\partial}_\infty = \bar{\partial} + \tfrac{1}{4}\pi\mu(F)nr^{2(n-1)}Iz\,\mathrm{d}\bar{z}.$$

By the discussion in § 5 *a* all, except possibly $\bar{\partial}_\infty$, are in the same orbit in $\mathcal{A}(\tilde{F})/\mathcal{G}^{\mathrm{c}}(\tilde{F})$ and this orbit is stable by theorem 3.1.

Now (modulo taking subsequences and L_2^2 changes of gauge) $A_j \rightharpoonup A_\infty$, weakly in the sense of L_1^2-connexions with variable model singularity. Near the marked point we have $\bar{\partial}_\infty - \bar{\partial}_j = \frac{1}{4}\pi\{\mu(F) - \mu(F_j)\}nr^{2(n-1)}Iz\,\mathrm{d}\bar{z}$ so that the weak convergence is preserved (using any Hermitian metric on $\tilde{F}$) and we can assume $\bar{\partial}_j \rightharpoonup \bar{\partial}_\infty$, weakly in L_1^2.

Exactly as in the proof of lemma 1 of Donaldson (1983) we obtain a non-zero weak limit L_2^2 endomorphism, g_∞, intertwining $\bar{\partial}_0$ and $\bar{\partial}_\infty$. Now, as usual, the problem is to show that g_∞ is an automorphism but in our case this is a triviality. Because A_∞ is Yang–Mills the corresponding parabolic structure on F and holomorphic structure $\bar{\partial}_\infty$ on $\tilde{F}$ are semi-stable. So g_∞ intertwines a stable and a semi-stable holomorphic structure on $\tilde{F}$ and so, by a standard property of (semi-)stable V-bundles, g_∞ must be an automorphism. It follows, again from § 5 *a*, that A_0 and A_∞ also lie in the same $(\mathcal{G}^{\mathrm{c}})^2$-orbit, as required. This concludes the proof of theorem 5.7.

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Remark. It is possible to show that the limit connexion lies in the same orbit by performing the analysis of lemma 1 of Donaldson (1983) on the singular connexions A_j, directly in the unitary frame on $\tilde{F}$. (Notice then that the intertwining automorphisms change the model singularity and so they must be *less* regular than L_2^2 in this frame, by corollary 4.5.) We will use this approach in the proof of theorem 6.1.

Being diagonal near the marked point, the connexion constructed in theorem 5.7 clearly behaves well with respect to the unitary clutching map t, of (6), and so carries back to a (Yang–Mills) weighted unitary connexion on F. As the singular $\overline{\partial}$-operators on $\tilde{F}$ are in the same $(\mathcal{G}^{\mathrm{c}})^2$-orbit, our discussion in § 5 *a* implies that on the weighted bundle the two weighted connexions are carried one to the other by a weighted automorphism.

Our main results (theorem 2.1 and proposition 2.2) follow, uniqueness being a simple consequence of that in theorem 5.7.

6. Parabolic Higgs bundles

In this section we apply the ideas which we have developed for dealing with the Narasimhan–Seshadri theorem on parabolic bundles to the analogous theorem for parabolic bundles with a 'Higgs field' interaction term. Our prototype theorem in this case is due to Hitchin (1987, theorem 4.3) and we obtain results similar to those of Simpson (1988, 1990) (see also Konno 1992).

First we discuss the correspondence between Higgs V-bundles and parabolic Higgs bundles with rational weights. Let $\mathcal{E} \to \tilde{M}$ be a holomorphic V-bundle and let $\phi : \mathcal{E} \to \mathcal{E} \otimes K_V$ be a holomorphic map, where K_V denotes the canonical V-bundle. Then ϕ is a *Higgs field* on $\mathcal{E}$ and the pair $(\mathcal{E}, \phi)$ is a *Higgs V-bundle*. As usual, stability of Higgs V-bundles is defined using only the ϕ-invariant sub-V-bundles.

If $\mathcal{F}$ is the parabolic bundle associated to $\mathcal{E}$ then we want to check that ϕ carries across to a parabolic Higgs field on $\mathcal{F}$: since a Higgs field is a $(1,0)$-form valued endomorphism a little care needs to be taken. Using a holomorphic trivialization with coordinates which respect the V-structure, about the marked point we have a Taylor series expansion analogous to (5):

$$\phi_{ij}\mathrm{d}z = \begin{cases} z^{x_i - x_j - 1}\phi'_{ij}(z^n)\,\mathrm{d}z & \text{if } x_i > x_j, \\ z^{n + x_i - x_j - 1}\phi'_{ij}(z^n)\,\mathrm{d}z & \text{if } x_i \leqslant x_j, \end{cases} \tag{24}$$

where the ϕ'_{ij} are holomorphic functions. To transfer this across to $\mathcal{F}$ we simply conjugate by the clutching function s defined by (4); at the marked point itself we will have a singularity which may or may not be removeable. We obtain

$$\begin{aligned} \{s\phi s^{-1}\}_{ij}\,\mathrm{d}z &= z^{x_j - x_i}\phi_{ij}\,\mathrm{d}z \\ &= \begin{cases} \phi'_{ij}(w)\mathrm{d}w/nw & \text{if } x_i > x_j, \\ \phi'_{ij}(w)\mathrm{d}w/n & \text{if } x_i \leqslant x_j, \end{cases} \end{aligned} \tag{25}$$

which is precisely a parabolic Higgs field.

This defines a correspondence between Higgs V-bundles and parabolic Higgs bundles (with appropriate parabolic weights). One can easily check that the invariant subbundles and hence the stable objects correspond. One therefore has an analogue of the result of Furuta & Steer (theorem 3.1) and also of the results of Mehta & Seshadri (1980, § 2) discussed in § 3 *b*. Further details can be found in Nasatyr (1991) and Nasatyr & Steer (1995); the latter particularly for results on the existence of stable Higgs V-bundles.

Now suppose that F is a Hermitian weighted bundle equipped with a parabolic Higgs bundle structure. Under the unitary clutching construction of § 4 *b* we obtain a Hermitian V-bundle $\tilde{F}$ with an initial singular connexion A_0 (compare corollary 4.6). From the parabolic Higgs field, exactly as in lemma 4.1, we obtain an $L^2_{k_0-1}$ map from $\tilde{F}$ to $\tilde{F}\otimes K_V$ which is a zero of the singular $\overline{\partial}$-operator induced by the initial singular connexion. If A is an L^2_1 singular connexion and $\phi : \tilde{F} \to \tilde{F}\otimes K_V$ an L^2_1 zero of the induced singular $\overline{\partial}$-operator then the pair (A, ϕ) is called an L^2_1 *singular Higgs pair*. A $\underline{\kappa}$-singular Higgs pair is said to be *Yang–Mills–Higgs* if $F_A + [\phi, \phi^*] = -2\pi \mathrm{i}\mu(\underline{\kappa})(*1)I$.

We prove the following theorem, the analogue of theorem 5.7 for L^2_1 singular Higgs pairs.

Theorem 6.1. *Let F be a Hermitian weighted bundle and $\tilde{F} = \tilde{F}_{n;x_1,\ldots,x_r}$ a k_0-approximation to F. Let $(\mathcal{F}, \phi)$ be a stable parabolic Higgs bundle structure on F and let (A_0, ϕ_0) be the corresponding initial L^2_1 singular Higgs pair on $\tilde{F}$. Then, provided k_0 is sufficiently large, there exists $g \in (\mathcal{G}^{\mathrm{c}})^2(\tilde{F})$, smooth away from the marked point, such that $g(A_0, \phi_0)$ is Yang–Mills–Higgs (with respect to a given orbifold Riemannian metric) and smooth away from the marked point. Moreover, g is unique up to the action of $\mathcal{G}^2$.*

Proof. We adopt the notation of § 5 *c*, taking a sequence of rational approximations $x_i^{(j)}/n_j$, $j = 1, 2, \ldots$, to the weights λ_i. The orbifolds $\tilde{M}_j$ can always be supposed to have negative orbifold Euler characteristic in any case where stable parabolic Higgs bundles exist by taking $n = n_1$ large enough. An orbifold of negative orbifold Euler characteristic is always finitely orbifold covered by a smooth Riemann surface (Fox 1952). Applying the equivariant existence result of Simpson (1988), we obtain an element $g_j \in (\mathcal{G}^{\mathrm{c}})^2(\tilde{F}_j)$ taking the initial data on $\tilde{F}_j \to \tilde{M}_j$ to a smooth Yang–Mills–Higgs pair. Moreover, the proof of Simpson's theorem (particularly his propositions 5.3, 6.6 and lemma 7.1) shows that

$$\sup_X |g_j| \leqslant N_j.$$

The constant N_j *a priori* depends on j but closer examination of the proof shows that it can be taken to depend only on $c_1(\tilde{F}_j)$ and certain elliptic and Sobolev constants: these are all bounded in j and so there is a uniform constant, N.

Transferring the solutions to $\tilde{F} = \tilde{F}_1$, we obtain a sequence of singular Yang–Mills–Higgs pairs (A_j, ϕ_j) on $\tilde{F}$, A_j having $(\underline{x} - \underline{x}^{(j)}n/n_j)$-singularity. We need some regularity for these in order to apply proposition 4.10 but we can't apply the arguments of § 5 *b* to obtain specific local forms at the marked point. Instead we note that the result of conjugating a smooth matrix by $t^{-1}h_jt_j$ is L^p_1 for any $1 < p < 2$. Hence the A_js and ϕ_js are L^p_1.

On $\tilde{F}$ we can still apply the bound $\sup_{X\setminus\{p\}} |g_j| \leqslant N$, as $t^{-1}h_jt_j$ is unitary, to conclude that the L^2 norms of the ϕ_js obey a uniform upper bound (indeed, the same is true for the L^{2p} norms). Arguing exactly as in Hitchin's proof we see that the L^2-curvature of the sequence $\{A_j\}$ cannot concentrate and so the same is true *a fortiori* for the L^p curvature. (Here we need to apply a Weitzenböck formula and so we need to note that all the terms involved are L^2 even though the A_js and ϕ_js are only L^p_1, since conjugation by $t^{-1}h_j t_j$ preserves L^2. We also need, as above, that $\tilde{M}$ has negative orbifold Euler characteristic.)

Hence we can apply a version of proposition 4.10 for L^p_1 singular connexions (which holds if p is sufficiently close to 2; see the third remark following proposition 4.9) to conclude that (modulo L^p_2 gauge transformations and after passing to a subsequence and relabelling) there is a subsequence with a weak limit A_∞ in L^p_1, with $\underline{\epsilon}$-singularity. Since we have elliptic equations $\bar{\partial}_{A_j}\phi_j = 0$ with L^p_1 coefficients and L^{2p} bounds on the A_js (as $L^p_1 \hookrightarrow L^{2p}$ is compact) and the ϕ_js, we can also conclude that there is a weak limit ϕ_∞ in L^p_1. By continuity (A_∞, ϕ_∞) still satisfies the Yang–Mills–Higgs equations.

Now we can apply the argument of lemma 1 of Donaldson (1983) to conclude that there is a weak limit L^p_1-endomorphism g intertwining (A_0, ϕ_0) and (A_∞, ϕ_∞). Exactly as in § 5 *c*, we conclude that g is in fact an automorphism.

By local gauge-fixing and bootstrapping (Parker 1982, theorem 5.3) we can suppose that the pair (A_∞, ϕ_∞) is smooth everywhere, except at the marked point, where it is, say, L^p_2 (modulo the model singularity) and hence g is L^p_3, which implies L^2_2. ■

Now we carry the result on the auxiliary V-bundle $\tilde{F}$ (i.e. theorem 6.1) back to the weighted bundle F. Ideally, we might hope for a result analogous to theorem 2.1, which gives the existence of a (*smooth*) weighted automorphism solving the problem. However, the results of Simpson (1990) suggest that this is too much to hope for in general; in this respect the case of parabolic Higgs bundles is certainly deeper than that of parabolic bundles. However, we *are* able to prove the existence of a weighted automorphism which is smooth away from the marked point and C^1 everywhere and hence theorem 2.3.

Let $\tilde{g}_0$ be as in corollary 4.6 (note also the remark following that result). Either by the Gram–Schmidt procedure or by an implicit function theorem argument we can write $g\tilde{g}_0^{-1} = hg'$ with h and g' both L^2_2, h unitary and g' upper-triangular. Since h is unitary there is thus a gauge in which, locally, the singular connexion of the Yang–Mills–Higgs singular pair is given by acting on $d - \Lambda$ by an upper-triangular automorphism and thus the associated singular $\bar{\partial}$-operator is also upper-triangular.

The point of gauge-fixing so that the automorphism and singular $\bar{\partial}$-operator are upper-triangular is, of course, that these will then behave well with respect to the clutching map t: in fact the result of conjugating an upper-triangular matrix which is continuous in z by t is clearly continuous in w. Moreover, since the resulting weighted automorphism is a C^0 solution of a first-order elliptic differential equation with C^0 coefficients it is certainly C^1.

Thus we have obtained as a solution a weighted automorphism which is smooth away from the marked point and C^1 (in w) at it. We might ask whether it is possible, in fact, to gain any greater regularity at the marked point: unfortunately the answer is no. Because $\partial/\partial w = (nz^{n-1})^{-1}(\partial/\partial z)$ and we have, at most, control of the first $k_0 - 1$ derivatives in z (§ 5 *a*), with $k_0 - 1 < n/2$ by (9), we clearly cannot hope to control the second derivative in w.

Notice that even if we allow weighted automorphisms which are only C^1 at the marked point then lemma 4.1 still holds. Thus, for instance, uniqueness on the weighted bundle follows from uniqueness on the V-bundle and we have proved theorem 2.3.

We can rephrase the theorem as follows (at the expense of broadening our concept of a weighted Hermitian metric). We fix the initial weighted Higgs pair and consider the problem of varying the weighted Hermitian metric so that the given pair becomes Yang–Mills–Higgs; this is entirely equivalent to the problem we have been considering up to this point. Then theorem 2.3 simply says that we can find a suitable metric which locally has the form,

$$g^* \begin{pmatrix} |w|^{2\lambda_1} & & 0 \\ & \ddots & \\ 0 & & |w|^{2\lambda_l} \end{pmatrix} g,$$

in coordinates which respect the flag structure, where, unlike in (3), g is now only required to be C^1 at the marked point. In this form it is apparent that what we have is an example of a Hermitian metric over the complement of the marked points which is 'harmonic' and has 'tame' growth at the marked points (Simpson 1990).

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